A Formalism for Quantum Games and an Application

PSU Analysis Seminar
December 2013

Steven A. Bleiler
Portland State University
Game Theory

Given
- a set of players \{1, 2, ..., n\},
- a set \(S_i\) of pure strategies for each player
- a set \(\Omega_i\) of possible outcomes or payoffs for each player
- a game \(G\) is a function

\[
G : S_1 \times S_2 \times \cdots \times S_n \rightarrow \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n
\]

which assigns to each n-tuple of strategic choices an outcome to each player.
• Without loss of generality, the $\Omega_i$’s are all copies of $\mathbb{R}$.

• A strategy profile is an n-tuple $(s_1, s_2, \ldots, s_n)$ where each $s_i \in S_i$ is a pure strategy chosen by player $i$. 
When playing a game

Rational players seek to identify

- a strategy which guarantees them an outcome having maximal utility

- a *security strategy*: a strategy that guarantees a specific lower bound to the utility received

- for a fixed $n-1$ tuple of opponents' strategies, a *best reply*: a choice $s_i \in S_i$ of strategy that delivers a utility at least as great of any other strategic choice.
In symbols, $s_i$ is a best reply to the $n$-1 tuple

$$(s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$$

of opponents' strategies if and only if

$$G(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) \geq G(s_1, ..., s_{i-1}, s, s, ..., s_n)$$

for all $s \in S_i$. 
Nash Equilibrium

A Nash equilibrium (also called a solution or just an equilibrium) is a strategy profile \((s_1, s_2, \ldots, s_n)\) such that each \(s_i\) is a best reply to the \(n-1\) tuple of opponents' strategies.

The identification of Nash equilibria is a fundamental goal in the theory of games.
The payoff matrix for this 2x2 game is

\[
\begin{array}{c|cc}
   & t_1 & t_2 \\
\hline
s_1 & (3,3) & (0,5) \\
\hline
s_2 & (5,0) & (1,1) \\
\end{array}
\]

\[S_1 = \{s_1, s_2\}; \quad S_2 = \{t_1, t_2\}; \quad \Omega_1 = \Omega_2 = \mathbb{R}\]

\[G: \{s_1, s_2\} \times \{t_1, t_2\} \to \mathbb{R}^2\]

For example: \(G(s_1, t_2) = (0,5)\)
Note that $G(s_2, t) \geq G(s_1, t)$ for all $t \in S_2$.

Similarly, $G(s, t_2) \geq G(s, t_1)$ for all $s \in S_1$.

Therefore, $s_2$ is a best reply to $t_2$ and $t_2$ is a best reply to $s_2$.

Hence, $(s_2, t_2)$ is the unique pure strategy Nash equilibrium of $G$. 

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>(3, 3)</td>
<td>(0, 5)</td>
</tr>
<tr>
<td>$s_2$</td>
<td>(5, 0)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>
A Caution

A game need not have a pure strategy equilibrium. An example is the game known as Simplified Poker, whose payoffs are:

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$(5/4, -5/4)$</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$(0, 0)$</td>
<td>$(5/2, -5/2)$</td>
</tr>
</tbody>
</table>

The standard way to proceed is to extend $G$ to a larger game $G^{mix}$. 
Mixing Strategies

- Each player plays each of his or her pure strategies with a specific probability, determined by a probability distribution over his or her pure strategies, a so-called *mixed strategy*.
- Given a profile of such strategies we form the product distribution over the set of pure strategy profiles.
- Applying $G$, we obtain a probability distribution over $ImG$.
- Applying the expectation operator $\varepsilon$ to this probability distribution, we then obtain the expected outcome for this strategic profile.
- Assigning the expected outcome to each mixed strategy profile gives $G^{mix}$.
- Denote the set of probability distributions over $S_i$ by $\Delta(S_i)$.

- The space $S_i$ embeds in $\Delta(S_i)$ by considering each element of $S_i$ as given by the probability distribution which assigns 1 to that element and 0 to the others.

- When $S_i$ is finite, $\Delta(S_i)$ is just the set of real convex linear combinations of the elements of $S_i$. 

Extending $G$ to a new game $G^{mix}$

The extension of $G$ by $G^{mix}$ is indicated in the following commutative diagram
$G^\text{mix}$ for Simplified Poker

- $\Delta(S_1) = \{ ps_1 + (1-p)s_2 \mid 0 \leq p \leq 1 \} \equiv [0, 1]$
- $\Delta(S_2) = \{ (1-q)t_1 + qt_2 \mid 0 \leq q \leq 1 \} \equiv [0, 1]$
- $G^{\text{mix}}(p,q) = (5p/4 + 5q/2 - 15pq/4, -5p/4 - 5q/2 + 15pq/4)$
- $G^{\text{mix}}(2/3,1/3) = (5/6, -5/6)$
- A direct calculation shows that $(2/3, 1/3)$ is the unique mixed strategy equilibrium of $G^{\text{mix}}$. 
Nash’s Theorem (1957)

- For a finite game $G$, there is always a Nash equilibrium in $G^{\text{mix}}$. 
More extensions of G

One begins by observing that the function

$$\prod_{i=1}^{n} \Delta(S_i) \rightarrow \Delta(\text{Im}G)$$

is not onto. If player I plays his first strategy with probability $p$, and player II plays her second strategy with probability $q$, the resulting probability distribution over the $\text{Im}G$ is given by

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$p(1-q)$</td>
<td>$pq$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>$(1-p)(1-q)$</td>
<td>$(1-p)q$</td>
</tr>
</tbody>
</table>
An easy exercise now shows that the element of \( \Delta(\text{Im } G) \) represented by

\[
\begin{array}{c|cc}
&s_1&t_1&t_2 \\
\hline
s_1&1/2&0 \\
\hline
s_2&0&1/2 \\
\end{array}
\]

is not realizable by any choice of \( p \) and \( q \).

Mediated classical communication addresses this issue.
Suppose a referee can be hired for a negligible cost.

For a given $d$ in $\Delta(\text{Im} G)$ the referee enforces $d$ by performing a random act with probability distribution $d$. This selects an outcome of $G$.

Referee now communicates to the players their and only their strategic choice that produces this outcome.
Note that the players are no longer playing the game $G$ but in fact a larger game, described here for 2 x 2 games, but easily generalized to games with more players and strategies.
Suppose the players’ strategy spaces for $G$ are represented by the set \{A, B\}.

Strategy spaces for $G_{com}^d$ are then represented by \{A', B', C', D'\}, where C' is always co-operate with referee, D' is always deviate, A' is play A always, and B' is play B always.
Note:

- If both players choose to play C’ the outcome of $G_d^{com}$ is exactly the expected outcome under $d$.

- The game $G_d^{com}$ extends $G$ as there is an embedding $f_i$ for each player from \{A, B\} to $T=\{A’, B’, C’, D’\}$ with $f_i(A)=A’$ and $f_i(B)=B’$ and such that

$$G = G_d^{com} \circ \prod_{i=1}^{2} f_i$$
Extending $G$

- Classical mediated communication thus gives a *family* of extensions of $G$ indexed by $\Delta(\text{Im } G)$. 
Correlated Equilibrium

- Following Aumann, a *correlated equilibrium* for $G$ occurs whenever $(C',C')$ is a Nash equilibrium in $G_{d}^{\text{com}}$.
- This says the agreement to follow the referees recommendation is *self policing*, which means that no player has an incentive to deviate from the referees recommendation.
- The use of correlated equilibria may or may not improve the lot of the players.
A classic example of correlated equilibrium improving the players’ lot is given by the following variant of Chicken:

<table>
<thead>
<tr>
<th></th>
<th>$t_1$</th>
<th>$t_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>(2, 2)</td>
<td>(0, 3)</td>
</tr>
<tr>
<td>$s_2$</td>
<td>(3, 0)</td>
<td>(−1, −1)</td>
</tr>
</tbody>
</table>
This game has two pure strategy equilibria and one mixed strategy equilibrium where each player plays their individual strategies with equal probabilities.

The mixed strategy equilibrium pays out 1 to each player.

Even without a referee any real convex linear combination of these three outcomes forms a self policing agreement between the players.
For example, players could jointly observe a fair coin and agree to play \((s_1, t_2)\) if heads and \((s_2, t_1)\) if tails. Outcome here is \((1.5, 1.5) > (1,1)\).

Even better and outside these linear combinations is the correlated equilibrium arising from the pay-out \[
\frac{1}{3}(2,2) + \frac{1}{3}(0,3) + \frac{1}{3}(3,0)\] which has pay-out \[
\left(\frac{5}{3}, \frac{5}{3}\right)\].
Prisoner’s Dilemma and Simplified Poker

- In these games mediated communication does not improve the lot of the players.
- In Prisoner’s Dilemma, because of the strong domination, players always have an incentive to deviate if $d$ assigns a non-zero probability to any outcome other than (1,1).
- For Simplified Poker any deviation from the equilibrium strategies $(2/3, 1/3)$ is fully exploitable by the other player and hence an incentive to deviate from any other potential correlated equilibrium strategy.
Quantization

- Of particular interest to us are extensions that use the generalized notion of probability distribution given by quantum superposition.
- For finite sets, a quantum superposition is given by a complex projective linear combination as opposed to a probability distribution which is given by a real convex linear combination.
- We will denote the set of quantum superpositions over a set \( X \) by \( QS(X) \).
• For example, if \( X = \{x, y\} \) then a complex \textit{projective} linear combination has form

\[
\left\{ x \alpha + y \beta \right\} \lambda \equiv x \alpha + y \beta \quad \text{for any nonzero complex number} \ \lambda, \ \text{called a} \ \textit{phase}.
\]

• These become quantum superpositions once we identify \( x \) and \( y \) with an orthogonal basis of a quantum system.
Quantum measurement

When the underlying complex vector space is a Hilbert space, we can assign a real length $|\ |$ to each complex linear combination of basis vectors and assign to each component the ratio of the square of its coefficient to the square of the length of the combination.

This allows us to obtain a real convex linear combination of our vectors via projection onto a basis. This is called *quantum measurement with respect to this basis* and denoted $q_{\text{meas}}$.

For example, the real convex linear combination produced from $x\alpha + y\beta$ is

\[
q_{\text{meas}}(x\alpha + y\beta) \in \mathbb{R} = \frac{|\beta|^2}{|\alpha|^2} + \frac{|\alpha|^2}{|\beta|^2}
\]
Game Quantization

For a finite game $G$, we extend $G$ to a new game $G^Q$ as follows.

Given

- A collection of nonempty sets $Q_1, Q_2, \ldots, Q_n$

- A *protocol*, that is, a function $Q : \prod_{i=1}^n Q_i \to QS(\text{Im}G)$

- Applying $q_{meas}$ then gives a probability distribution over $\text{Im}G$.

- We can now form a new game $G^Q$ by applying the expectation operator $\varepsilon$. 
Proper and Complete Quantizations

- The $Q_i$’s are the pure quantum strategy sets.

- If there exist embeddings

  \[ e'_i : S_i \rightarrow Q_i \quad \text{with} \quad G^Q \circ \Pi e'_i = G \]

  then $G^Q$ is said to be a \textit{proper} quantization of $G$.

- If there exist embeddings

  \[ e''_i : \Delta(S_i) \rightarrow Q_i \quad \text{with} \quad G^Q \circ \Pi e''_i \circ \Pi e_i = G \]

  then $G^Q$ is said to be a \textit{complete} quantization of $G$. 
In pictures:

• A complete quantization is a proper quantization.
• Finding a proper quantization of a game $G$ is just the usual mathematical problem of extending a function.
Mixed quantizations

Nothing prevents us from having $G^Q$ play the role of $G$ in the classical mixtures. This gives the \textit{mixed quantization of $G$ by the protocol $Q$}. Call this game $G^{mQ}$.
EWL protocols a.k.a Mediated Quantum Communication

- Eisert, Wilkens, and Lewenstein (EWL) put forward a specific family of protocols $Q^I$ for the quantization of 2 player, 2 strategy games.
- The EWL protocols require the game $G$ to be played with a referee who communicates with the players through a quantum channel, i.e., players and the referee can send superpositions of the two states $|0\rangle$ and $|1\rangle$ of a classical bit, a so-called quantum bit or qubit for short.
• Each player is sent a qubit (a two-state quantum system or a unit vector in a projective Hilbert space) by the referee. The qubits sent to the players are in a joint initial state, i.e., a quantum superposition of the joint states.

• For two players, this has form

\[
\begin{pmatrix}
|00\rangle & |01\rangle & |10\rangle & |11\rangle
\end{pmatrix}
\]

• Each choice of the quadruple \((\alpha, \beta, \gamma, \delta)\) gives an initial state \(I\) and this initial state produces in turn a specific protocol \(Q^I\).

• Players act on their qubit via elements of \(SU(2)\) and send back the final product. In particular, the two classical pure strategies available to the players are represented by \(N\) (no flip) and \(F\) (flip).

• The referee expresses the final state in a specific basis of \(QS(ImG)\) and then assigns via \(q_{meas}\) the appropriate expected-payoffs to the players.
The Landsburg Classification

- S. Landsburg examines the quantum game arising from the EWL protocol applied to 2 player, 2 strategy games and corresponding to the initial state

$$I = \frac{1}{\sqrt{2}} \left( |00\rangle + |11\rangle \right)$$

- For this initial state the map

$$q_{meas} \circ Q^I : \prod Q_i \rightarrow \Delta(\text{Im} G)$$

is onto.

- Landsburg determines the potential Nash equilibria in both the games $G^Q$ and $G^{mQ}$. In particular, there never exists pure quantum equilibria; that is, there are no Nash equilibria in the game $G^Q$. 
• However, there is always at least one Nash equilibria amongst the mixed quantum strategy profiles.

• In particular, when each player uses the uniform probability distribution over his or her pure quantum strategies, each player is guaranteed to receive at least the average of the four classical payouts.
A Fundamental Question

- Is the Nash equilibria in the quantized game truly new?

- Specifically, is the probability distribution that arises from the mixed quantum equilibrium different from that of a classical correlated equilibrium?
The Answer

- The mixed quantum strategy equilibrium requires the players to use the uniform probability distribution over his or her choice of pure quantum strategy thus producing the uniform distribution over the four classical payoffs.

- For the Prisoner’s Dilemma this assigns a non-zero probability other than (1,1) and hence is not a classical correlated equilibrium.

- Note: the mixed quantum equilibrium increases the payoffs to both players.
• An even more amazing result holds for the 0-sum game of Simplified Poker, where the uniformly mixed quantum equilibrium out performs the classical mixed strategy equilibrium payoffs for player I, yet is still a security strategy for player I against which player II has no recourse.
Conclusion

- Our discussion has shown that for Prisoner’s Dilemma and Simplified Poker quantization indeed holds something new for Game Theory.

- Several other controversies in quantum games can be similarly resolved via the formalism presented here and new game theoretic interpretations of certain problems from both quantum computation and quantum logic synthesis can also be eliminated.

- Watch this space for details.
The preprint for today’s presentation is currently available at 


Thanks, and have a great day!