Some Algebras of Composition Operators

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1 *Composition operators*

We'll be studying an intriguing class of Hilbert-space operators connected with analytic function theory. Our setting is the open unit disc $\mathbb{U}$ of the complex plane, upon which lives the "Hardy Space" H^2, which consists of all functions f analytic on $\mathbb{U}$ whose Maclaurin-coefficient sequence is square summable. More precisely

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \quad \text{with} \quad \|f\|^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\,.$$

The second sum above defines a norm on H^2 that makes it into a Hilbert space isometrically isomorphic to the sequence space ℓ^2.

We'll be particularly interested in the class of linear transformations on H^2 induced by analytic functions φ that take $\mathbb{U}$ into (possibly onto) itself. These are the "holomorphic selfmaps" of $\mathbb{U}$.

"Holomorphic" means "analytic".

Any such map φ induces, by means of the formula $C_\varphi f = f \circ \varphi$, a linear transformation C_φ on the vector space of *all* complex-valued functions that are analytic on $\mathbb{U}$. One of the miracles of analytic function theory is:

The Littlewood Subordination Theorem ([8], 1925). *For each holomorphic selfmap φ of $\mathbb{U}$, the induced composition operator maps H^2 into itself, and its restriction to H^2 is a bounded linear operator on that space.*[1]

[1] To better appreciate the depth of Littlewood's Theorem, imagine trying to prove directly that C_φ maps H^2 into itself for the special case $\varphi(z) = \frac{1+z}{2}$. For $f \in H^2$ this involves substituting $\varphi(z)$ for z in the Maclaurin series expansion of f, using the binomial theorem to expand the resulting powers of $\varphi(z)$, and interchanging the order of summation to get the Maclaurin coefficients of $f(\frac{1+z}{2})$. It's not obvious that this new Maclaurin-coefficient sequence, each term of which is a sum involving both binomial coefficients and the original Maclaurin coefficients of f, has the desired square-summability. For a general holomorphic selfmap φ of the unit disc, the situation is exponentially worse.

This theorem of Littlewood poses the challenge of relating the operator-theoretic properties of C_φ (on H^2) with the function-theoretic properties of its "inducing map" φ. For example:

- What is the *norm* of C_φ?
- What is the *spectrum* of C_φ?
- What is the *commutant* of C_φ?
- What is the *double commutant* of C_φ?

We'll devote the rest of this note to studying these questions when the mapping φ is about as simple as it can get: $\varphi(z) = \frac{1+z}{2}$. Even this special case will lead to fascinating mathematics, and to problems that appear, even now, to be open.

2 *Norm and spectrum*

From now on, $\varphi(z)$ will always be $\frac{1+z}{2}$, and we'll allow ourselves to abuse notation by writing C_φ as $C_{\frac{1+z}{2}}$. We'll take Littlewood's Subordination Theorem for granted.[2]

[2] For a couple of different proofs see, e.g., [4, §2.6, pp. 28–29] and [11, pp. 13–17].

2.1 *Norm*

Once you know that a Banach-space operator is bounded, it makes sense to ask for its *norm*, i.e., the supremum of the lengths of the images of unit vectors. For many operators this can be surprisingly difficult. Composition operators, even very simple ones, illustrate this.

Theorem 1. (C. Cowen, [3], 1988).[3] *If* $|s| + |t| \leq 1$, *then:*

$$\|C_{sz+t}\| = \sqrt{\frac{2}{1 + \Delta + \sqrt{(1-\Delta)^2 - 4|t|^2}}},$$

where $\Delta := |s|^2 - |t|^2$.

Corollary 2. $\|C_{\frac{1+z}{2}}\| = \sqrt{2}$.

[3] Theorem 1 tells us that if s and t are rational, then $\|C_{sz+t}\|$ is an algebraic number. This is not always the case for composition operators induced by linear fractional selfmaps of the unit disc. Recently Bourdon et al. [1] have shown that, for example, $\|C_{\frac{1}{2-z}}\|$ is transcendental!

2.2 *Spectrum*

For a linear transformation T on a complex Hilbert (or Banach) space, the *spectrum*, $\sigma(T)$, is the set of complex numbers λ for which $T - \lambda I$ is *not invertible*. Every eigenvalue belongs to the spectrum, and if our ambient vector space has finite dimension, that's all there is. However the infinite dimensional situation is different. The spectrum of a bounded operator on a Hilbert (or Banach) space is always compact, and never empty,[4] but in infinitely many dimensions, there may be no eigenvalues (example: The Volterra operator on $L^2([0,1])$).

[4] See [10], Ch. 18, pp. 356–359 (esp. Cor. 3) for an exposition of these matters—in the context of Banach algebras.

Nevertheless, if $|\lambda| > \|T\|$ then A simple geometric-series argument establishes an important bound on the size of the spectrum: if $|\lambda| > \|T\|$ then $T - \lambda I$ *is* invertible, with inverse given by the "Neumann series" $\sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n$, which converges in operator norm.

SUMMARY:

$$\sigma(T) \subset \|T\| \cdot \overline{\mathbb{U}} := \{\lambda \in \mathbb{C} \colon |\lambda| \leq \|T\|\}.$$

In particular,

$$\sigma\left(C_{\frac{1+z}{2}}\right) \subset \sqrt{2} \cdot \overline{\mathbb{U}}. \tag{2}$$

The spectrum of an operator contains all its *eigenvalues*. Now the operator $C_{\frac{1+z}{2}}$, as it acts on the space of all analytic functions on $\mathbb{U}$, has an obvious[5] collection of eigenvectors:

$$e_w(z) = (1-z)^w \qquad (w \in \mathbb{C}), \tag{3}$$

in which case

$$C_{\frac{1+z}{2}}(e_w) = 2^{-w} e_w\,. \tag{4}$$

We're interested in the restrictions of composition operators to the Hardy space H^2, so we need to restrict w so that the eigenfunction $e_w \in H^2$ for $C_{\frac{1+z}{2}}$ lies in H^2, i.e., we must demand that $\operatorname{Re} w > -\frac{1}{2}$. CONSEQUENCE: *The eigenvalues of $C_{\frac{1+z}{2}}$ fill the open disc $\sqrt{2}\cdot\mathbb{U}$.*

[5] What's "obvious" is Eqn. (4) below at the left. where we define $(1-z)^w$ to be $e^{w\,\mathrm{Log}\,(1-z)}$, with "Log" denoting the principal value of the complex logarithm. What needs to be proved is that e_w is analytic on $\mathbb{U}$, and—when we restrict attention to H^2—that $e_w \in H^2$ iff $\operatorname{Re} w > -1/2$. For more on this, see the Appendix at the end of this note.

Theorem 3. $\sigma(C_{\frac{1+z}{2}}) = \sqrt{2}\cdot\overline{\mathbb{U}}$.

Proof. We've just seen that

$$\sqrt{2}\cdot\mathbb{U} \subset \sigma(C_{\frac{1+z}{2}}),$$

and we know from (2) that $\sigma(C_{\frac{1+z}{2}}) \subset \sqrt{2}\cdot\overline{\mathbb{U}}$. The Theorem now follows from the fact that the spectrum of any bounded operator on a Hilbert (or Banach) space is compact. □

3 *Commutant*

The *commutant* of a Hilbert-space operator T is the set of bounded operators on that Hilbert space that commute with T. Notation: $\{T\}'$.

One checks easily that $\{T\}'$ is an *algebra* that contains the identity map (i.e., is "unital") and is *weak-operator closed*.[6] Commutativity is preserved by weak-operator convergence, so upon letting $\mathrm{alg}(T)^w$ denote the weak-operator closure of the unital algebra generated by T, we see immediately that

$$\mathrm{alg}(T)^w \subset \{T\}'$$

for each Hilbert-space operator T.

[6] To say a collection $\mathscr{S}$ of bounded linear operators on a Hilbert space H is *weak-operator closed* means that: if (S_α) is a net of operators in $\mathscr{S}$, and T is a bounded operator on H for which $\langle S_\alpha x, y\rangle \to \langle Tx, y\rangle$ for each pair x, y of vectors in H, then $T \in \mathscr{S}$.

Here's one reason why it's important to study the commutant of an operator:

Proposition 4. *If T is a bounded operator on H and M is an invariant*[7] *subspace for T, then so is AM for any $A \in \{T\}'$.*

[7] Meaning: $T(M) \subseteq M$.

We'll begin our study the commutant of C_φ (where $\varphi(z) = \frac{1+z}{2}$) with the following question:

Is $\{C_\varphi\}'$ equal to $\mathrm{alg}(C_\varphi)^w$?

In other words: *Is the commutant of $C_{\frac{1+z}{2}}$ "minimal"?*

3.1 *Some operators in* $\{C_\varphi\}'$

Here are two classes of operators that commute with C_φ:

(a) *Composition operators* C_{φ_r}, where φ_r is the affine selfmap of $\mathbb{U}$ defined by $\varphi_r(z) = rz + (1-r)$ for $0 < r \le 1$.

According to Theorem 1, $\|C_{\varphi_r}\| = \frac{1}{\sqrt{r}}$.

Note that our map φ is just $\varphi_{1/2}$, and that each φ_r has the "same DNA" as φ in that it fixes the points 1 and ∞, and maps $\mathbb{U}$ onto an open sub-disc of $\mathbb{U}$ whose boundary is tangent to the unit circle at the point 1. Note further that

$$\varphi_r \circ \varphi_s = \varphi_{rs} \quad \text{for} \quad 0 < r, s \le 1,$$

so all the maps φ_r commute with each other, and consequently so do all the composition operators they induce on H^2.

Consequence: $C_{\varphi_r} \in \{C_\varphi\}'$ for each $0 < r \le 1$.

(b) *Certain multiplication operators*. If b is a bounded analytic function on $\mathbb{U}$ then the *multiplier* M_b, defined by

$$(M_b f)(z) = b(z)f(z) \qquad (z \in \mathbb{U},\ f \in H^2)$$

is a bounded linear operator[8] on H^2 that has a particularly interesting connection with composition operators. To wit: for each holomorphic selfmap ψ of $\mathbb{U}$ we have, on H^2:

[8] This, too, needs to be proved. It follows easily upon observing that H^2 is the collection of functions f holomorphic on $\mathbb{U}$ for which

$$\sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta}|^2 < \infty$$

(in which case above supremum is the square of the norm introduced in the original definition of H^2).

$$C_\psi M_b = M_{b\circ\psi} C_\psi. \tag{5}$$

i.e., C_ψ *intertwines* M_b with $M_{b\circ\psi}$.

In particular:

M_b commutes with C_ψ whenever $b \circ \psi = b$.

In other words:

$M_b \in \{C_\psi\}'$ whenever b is a C_ψ-eigenvector for the eigenvalue "1".

Returning to $\varphi(z) = \frac{1+z}{2}$, we've seen in §2.2 that for $w \in \mathbb{C}$ the function $e_w\colon z \to (1-z)^w$ is an eigenvector of C_φ (now acting on the space of all analytic functions on $\mathbb{U}$) for the eigenvalue 2^{-w}. Thus $C_\varphi e_w = e_w$ iff $2^{-w} = 1$, and this happens iff $w = \frac{2\pi i}{\ln 2}k$ for some integer k. In summary:

Proposition 5. *For* $w \in \mathbb{C}$*:*

$$C_\varphi e_w = e_w \iff w = i\tau k,$$

where $\tau = \frac{2\pi}{\ln 2}$, *and* $k \in \mathbb{Z}$.

When $w \in \mathbb{C}$ is pure imaginary, the eigenfunction e_w is bounded on $\mathbb{U}$, so—as noted above—the multiplication operator M_{e_w} acts boundedly on H^2. Furthermore, when $w \in i\tau \cdot \mathbb{Z}$, Proposition 5 assures us that M_{e_w} commutes with C_φ.

Collecting the result of part (a) above with that of Proposition 5:

Proposition 6. *For $\varphi(z) = \frac{1+z}{2}$, the commutant of C_φ contains the operators C_{φ_r} for $0 < r \leq 1$, and M_w where $\tau = \frac{2\pi i}{\ln 2}$ and $w \in i\tau \cdot \mathbb{Z}$.*

Thanks to Proposition 6 we (finally!) have some definitive results for the commutant of C_φ, where $\varphi(z) = \frac{1+z}{2}$:

Theorem 7. *$\{C_\varphi\}'$ is not commutative,*

Proof. Let $M = M_{e_{i\tau}}$. Since $\varphi_r \circ e_{i\tau} = r^{i\tau} e_r$, we see from Eqn. (5) above that

$$C_{\varphi_r} M = r^{i\tau} M \qquad (0 < r \leq 1).$$

Thus C_{φ_r} commutes with M iff $r^{i\tau} = 1$. A little arithmetic shows that this happens iff $r = 2^{-k}$ for some positive integer k. In particular, $C_{\varphi_{1/3}}$ does not commute with M. □

Since $\mathrm{alg}(C_\varphi)^w$ inherits the commutativity of $\mathrm{alg}(C_\varphi)$, Theorem 7 implies:

Corollary 8. $\{C_\varphi\}' \neq \mathrm{alg}(C_\varphi)^w$.

I.e., the commutant of C_φ is not *minimal*.

4 *Double Commutant*

Having established that the commutant of C_φ is strictly larger than the weak-operator closure of the algebra generated by C_φ, it makes sense to ask: "How much larger?"

For a general bounded operator T on a Hilbert space H, the size of its commutant is connected with that of its *double commutant* $\{T\}''$: the set of all bounded operators on H that commute with everything in $\{T\}'$. The double commutant of T is again a weak-operator closed algebra, which (since T belongs to $\{T\}'$) contains $\mathrm{alg}(T)^w$. Thus

$$\mathrm{alg}(T)^w \subseteq \{T\}'' \subseteq \{T\}' \qquad (6)$$

and we expect that "larger commutants mean smaller double commutants."

Note also that $\{T\}''$ is *commutative*. Reason: If $A, B \in \{T\}''$ then by (6), they both commute with T. Being in $\{T\}''$, they commute with every thing in $\{T\}'$, hence with each other.

For example, if T is identity operator on H, then $\{I\}' = \mathscr{L}(H)$, so $\{I\}'' = \mathbb{C} \cdot I$. Just as for commutants, we'll call $\{T\}''$ *minimal* whenever it's equal to $\mathrm{alg}(T)^w$. Thus the double commutant of the identity operator is minimal. The most famous result in this direction is:

von Neumann's Double-Commutant Theorem. *If T is a self-adjoint operator on a Hilbert space, then $\{T\}''$ is minimal.* [14, 1929]

The question of minimal double commutants has attracted a lot of attention. For example, every *algebraic* operator on Hilbert space (in particular, every operator on a finite dimensional Hilbert space) has minimal double commutant [13 (1972)]. For further references on the minimal double-commutant problem, see [7].

4.1 Carter's Theorem

Now back to $C_\varphi = C_{\frac{1+z}{2}}$: Is *its* double commutant minimal? We've seen in §3.1 that each $T \in \{C_\varphi\}''$ commutes with both the composition operators induced by the semigroup of affine maps $\{\varphi_r\colon 0 < r \leq 1\}$ and the semigroup of multiplication operators M_b, where $b = e_{i\tau k}$ for $k \in \mathbb{Z}$.

Our first result supposes only that T commutes with the first of these semigroups of operators. Its statement will involve the open half-plane $\mathbb{P} := \{w \in \mathbb{C}\colon \operatorname{Re} w > -\frac{1}{2}\}$,and will show, in particular, that for each $w \in \mathbb{P}$ the eigenfunction e_w of C_φ is also an eigenfunction of any operator in $\{C_\varphi\}''$.

Theorem 9. (James Michael Carter [2, 2013]): *Suppose $T \in \mathscr{L}(H^2)$ commutes with each operator C_{φ_r} for $0 < r \leq 1$. Then there exists a bounded holomorphic function F_T on $\mathbb{P}$ such that*

$$Te_w = F_T(w)e_w \quad \textit{for each } w \in \mathbb{P}.$$

This leads to a refinement of the set containments (6).

Corollary 10. $\operatorname{alg}(C_{\frac{1+z}{2}})^w \subseteq \{C_{\frac{1+z}{2}}\}'' \subsetneq \{C_{\frac{1+z}{2}}\}'$.

Proof. We've already noted the non-strict set containments, and have seen in Theorem 7 that $\{C_{\frac{1+z}{2}}\}'$ is not commutative. □

Proof of Theorem 9. Fix $w \in \mathbb{P}$ and $T \in \{C_\varphi\}''$. Then for each $r \in (0,1]$ and $z \in \mathbb{U}$:

$$r^w(Te_w)(z) = (TC_{\varphi_r}e_w)(z) = C_{\varphi_r}Te_w(z) = (Te_w)(rz + (1-r)),$$

where the middle equality expresses the fact that T commutes with C_{φ_r}. Upon setting $f(z) = Te_w(z)$ so $f \in H^2$ we summarize the above computation:

$$f(rz + (1-r)) = r^w f(z) \qquad (z \in \mathbb{U} \quad \& \quad 0 < r \leq 1).$$

Now differentiate the above expression with respect to r:

$$f'(rz + (1-r))(z-1) = w\, r^{w-1} f(z) \qquad (0 < r \leq 1,\ z \in \mathbb{U}\ \&\ w \in \mathbb{P}),$$

and set $r = 1$, obtaining

$$f'(z)(z-1) = w\, f(z) \qquad (z \in \mathbb{U}\ \&\ w \in \mathbb{P}). \tag{7}$$

Upon solving the differential equation (7) for f we obtain the desired result:

$$f(z) = \text{const.}\,(1-z)^w = F_T(w)\,e_w(z) \qquad (w \in \mathbb{P}),$$

where $F_T(w)$ is the "constant of integration." □

Thanks to Carter's Theorem, the association of an operator $T \in \{C_\varphi\}''$ with the function F_T establishes a map, taking $\{C_\varphi\}''$ into $H^\infty(\mathbb{P})$. More specifically:

$$\text{(8)} \qquad \Phi(T)e_w = F_T(w)e_w \qquad (T \in \{C_\varphi\}'' \ \&\ w \in \mathbb{P}).$$

One checks easily that Φ is a Banach algebra isomorphism.[9]

In the next section we'll present work of Miguel Lacruz and his co-authors, who use the map Φ to establish the minimality of $\{C_\varphi\}''$.

[9] That Φ is one-to-one follows from the density of the linear span of the eigenfunctions $\{e_w : w \in \mathbb{P}\}$ in H^2.
Carter's argument (given above) depends only on the fact that the operator T commutes with each composition operator C_{φ_r} for $0 < r \leq 1$, and so the isomorphism Φ defined above really takes the weakly closed algebra $\mathscr{A}$ generated by these operators into $H^\infty(\mathbb{P})$. Carter goes further ([2], Chapter 5) to prove that $\Phi(\mathscr{A}) = H^\infty(\mathbb{P})$. This is not trivial!

4.2 *Work of Lacruz, Petrovic, Leon-Saavedra, and Rodriguez-Piazza*

First, some periodicity.

Proposition 11. *If $T \in \{C_\varphi\}''$ then $F_T(w+i\tau) = F_t(w)$ for each $w \in \mathbb{P}$.*

Proof. It's one line:

$$F_T(w)e_{w+i\tau} = M_f T e_w = T M_f e_w = T e_{w+i\tau} = F_T(w+i\tau)e_{w+i\tau}. \quad \square$$

Corollary 12. $\Phi(\{C_\varphi\}'') \subset H^\infty_\tau(\mathbb{P})$.

Now let $H^\infty_\tau(\mathbb{P})$ denote the subspace of $H^\infty(\mathbb{P})$ consisting of those functions f that are $i\tau$-periodic, i.e. for which $f(w+i\tau) = f(w)$ for each $w \in \mathbb{P}$.

Theorem 13. ([7, 2017]). $\Phi(\text{alg}(C_\varphi)^w) = H^\infty_\tau(\mathbb{P})$.

Proof. Fix $F \in H^\infty_\tau(\mathbb{P})$: *To find*: $T \in \text{alg}(C_\varphi)^w$ such that

$$\Phi(T) = F \quad \text{i.e.,} \quad Te_w = F(w)e_w \ \forall\, w \in \mathbb{P}.$$

Let $q(w) = 2^{-w}$. Then $q(w+i\tau) = q(w)$ for each $w \in \mathbb{P}$, from which it follows quickly that the function $f\colon \sqrt{2}\cdot\mathbb{U} \to \mathbb{C}$ defined by

$$f(2^{-w}) = F(w) \qquad (w \in \mathbb{P})$$

is holomorphic and bounded on $\sqrt{2}\cdot\mathbb{U}$.

By *Fejer's Theorem* there exists a sequence (p_n) of polynomials such that $p_n \to f$ pointwise on $\mathbb{D}$ and $\|p_n\|_\infty \leq \|f\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm over $\sqrt{2}\cdot\mathbb{U}$. Thanks to *von Neumann's Inequality* we know that for each index n:

$$\|p_n(C_\varphi)\| \leq \|p_n\|_\infty \leq \|f\|_\infty.$$

By Tychonov's Theorem, bounded subsets of $\mathscr{L}(H^2)$ are relatively compact in the weak operator topology, and by the separability of H^2 this topology is metrizable on bounded sets. Thus we replace the sequence (p_n) by a subsequence that converges weakly to some operator $T \in \text{alg}(C_\varphi)^w$. Consequently, for each $w \in \mathbb{P}$

$$Te_w = \lim_n p_n(C_\varphi)e_w = [\lim_n p_n(2^{-w})]e_w = f(2^{-w})e_w = F(w)e_w\,,$$

with the limit in the weak topology of H^2. Upon appealing once more to the density of span $\{e_w : w \in \mathbb{P}\}$ in H^2, we obtain the desired result: $\Phi(T) = F$. □

Now the *piece de résistance*.

Corollary 14. $\{C_{\frac{1+z}{2}}\}'' = \text{alg}(C_{\frac{1+z}{2}})^w$*, i.e.,* $\{C_{\frac{1+z}{2}}\}''$ *is minimal.*

Proof. Thanks to Theorem 9 and Proposition 11 we know that $\Phi(\{C_{\frac{1+z}{2}}\}'') \subset H^\infty_\tau(\mathbb{P})$. This along with Theorem 13 shows that:

$$H^\infty_\tau(\mathbb{P}) \supset \Phi(\{C_{\frac{1+z}{2}}\}'') \supset \Phi(\text{alg}(C_\varphi)^w) = H^\infty_\tau(\mathbb{P}),$$

so there is equality all the way through. Since Φ is one-to-one we conclude that $\{C_{\frac{1+z}{2}}\}'' = \text{alg}(C_{\frac{1+z}{2}})^w$. □

5 *Strong compactness*

So far we've seen that $\{C_\varphi\}'$ is "large" in the sense that $\{C_\varphi\}''$ is minimal (i.e., equal to $\text{alg}(C_\varphi^w)$), and not equal to $\{C_\varphi\}'$. In this section we'll discuss another way of measuring the size of an algebra of Hilbert-space operators.

Definition 15. *An algebra* $\mathscr{A} \subset \mathscr{L}(H)$ *is said to be "strongly compact" provided its unit ball is relatively compact in the* strong operator topology*, i.e., the topology of pointwise convergence on* H*.*

Whereas every bounded subset of $\mathscr{L}(H)$ is relatively compact in the *weak operator topology*, the same is not true for the strong topology. *Example*: The closed unit ball of $\mathscr{L}(H)$ itself, whenever $\dim H = \infty$.

This definition of strong compactness was initiated by Victor Lomonosov [9, 1980], who used it in his work on the Invariant Subspace Problem. The concept was introduced into the study of composition operators by Fernández-Valles and Lacruz in [5, 2012], and carried further by me in [12, 2013]. In this paper, I was able to show that for the operator C_φ we've been considering here, $\text{alg}(C_\varphi)^w$ *strongly compact*. However I was unable to decide if the same is true for $\{C_\varphi\}'$ (Conjecture: it's not).

We know that $\{C_\varphi\}'$ contains all the composition operators C_{φ_r} for $0 < r \leq 1$, and $M_{e_{ik\tau}}$ for $k \in \mathbb{Z}$. If any of these operators were to generate a weakly closed algebra that's not strongly compact, then the same would be true of $\{C_\varphi\}'$. Unfortunately, it's shown in [11] that each of these individually generated algebras *is* strongly compact.

6 Further results

Until now we've been focused on the composition operator C_φ where $\varphi(z) = \frac{1+z}{2}$. This is an example of a hyperbolic linear fractional self-map of the unit disc with attractive fixed point on the boundary and repulsive one outside the closed unit disc (in this case: at ∞). The table below summarizes known results for both hyperbolic ("H") and parabolic ("P) linear fractional self-maps of $\mathbb{U}$, both automorphic ("A"), i.e., taking the disc *onto* itself, and non-automorphic ("N"). The notation "SC" stands for "Strong Compactness." The results of this table all come from the papers [5, 6, 7,] and [12].

φ	FP's on	$\{C_\varphi\}'$ min'l?	$\{C_\varphi\}''$ min'l?	$\mathrm{alg}(C_\varphi)^w$ SC?	$\{C_\varphi\}'$ SC?
PA	$\partial\mathbb{U}$	N	N	Y	N
PNA	$\partial\mathbb{U}$	Y	Y	Y	Y
HA	$\partial\mathbb{U}$	N	N	Y	N
HNA	$\partial\mathbb{U}$ & $\mathbb{U}$	N	Y	N	N
	$\partial\mathbb{U}$ & $\mathbb{U}_e$	N	Y	Y	???

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