Introduction to the Cauchy-Riemann operator

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These notes introduce the reader to the non-homogeneous Cauchy-Riemann equation and its use in providing new proofs of some classical results in complex analysis. The story begins by showing how the classical Cauchy-Riemann equations, combined into the single complex equation $\bar{\partial} f = 0$, connect the concepts of real and complex differentiation. The scene then shifts to the *non-homogeneous* Cauchy-Riemann equation, which provides an audacious method for constructing analytic functions with prescribed behavior: first construct a "smooth" non-analytic prototype φ that otherwise exhibits the desired behavior. Then "correct" this prototype by adding an appropriate solution u of the non-homogeneous Cauchy-Riemann equation $\bar{\partial} u = -\bar{\partial}\varphi$ that preserves the desirable behavior behavior chosen to provide a function $f = \varphi + u$ that retains the desired behavior of the prototype, and is now, being a smooth function with $\bar{\partial} f = 0$, is analytic.

As an introductory application, we show how this method yields the classical Mittag-Leffler Theorem on the existence of meromorphic functions with prescribed singularities. As a further application, we unleash this method on the ring structure of the space of entire functions, where we use it to characterize the finitely generated ideals.

0 Notation

Throughout these notes, Ω denotes a nonempty open subset of the complex plane $\mathbb{C}$, with f a complex-valued function on Ω. We will write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, so $f = u + iv$, where u and v are real-valued functions on Ω.

We'll call f "smooth on Ω" whenever the partial derivatives of its real and imaginary parts exist and are continuous there, i.e., whenever the "real derivative" Df exists and is continuous on Ω (alternative terminology: f is C^1 on Ω, or $f \in C^1(\Omega)$).

We will frequently conflate $\mathbb{C}$ with $\mathbb{R}^2$, regarding Ω and $f(\Omega)$ as subsets of either (or both).

1 Differentiation: complex vs. real

We'll begin this section by defining the complex derivative as the usual difference-quotient limit, after which we'll show the necessity of the Cauchy-Riemann equations for the existence of this complex derivative. We'll discover the extent to which these equations are *sufficient* for complex differentiability by exploring the connection between complex differentiation for $\mathbb{C}$-valued functions f defined on subsets of the plane, and "real" differentiation for such f, but now viewed as $\mathbb{R}^2$-valued functions defined on subsets of $\mathbb{R}^2$.

1.1 Complex Differentiation I

Definition 1.1. *To say a function $f: \Omega \to \mathbb{C}$ is (complex-)* differentiable at the point $z_0 \in \Omega$ *means that the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

exists (finitely); we write $f'(z_0)$ for this limit, and call it the (complex) derivative *of f at z_0.*

To say that f is analytic on Ω *means that it is differentiable at each point of Ω.*

To say that "f is analytic *at z_0*" means that it is analytic in an *open neighborhood* of z_0.

If f is complex differentiable at z_0 then it follows upon writing successively $z = z_0 + t$ and $z = z_0 + it$ in (1) that

$$f'(z_0) = \lim_{t \to 0} \frac{f(z_0 + t) - f(z_0)}{t} = \lim_{t \to 0} \frac{f(z_0 + it) - f(z_0)}{it}$$

which we can summarize as:

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

Upon writing $f = u + iv$ where u and v are, respectively the real and imaginary parts of the function f, we infer from the above the existence of the partial derivatives at z_0 of both u and v with respect to x and y, and—now denoting these derivatives by subscripts—that at z_0:

$$u_x + iv_x = -i(u_y + iv_y).$$

Thus:

Proposition 1.2. *For complex differentiability of $f = u + iv$ at z_0 it's necessary that the partial derivatives of u and v with respect to x and y exist at z_0, and obey the* Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad v_x = -u_y \tag{2}$$

at z_0.

Question. *If the Cauchy-Riemann equations hold at z_0, is f complex-differentiable there?*

Exercise: Let $f(x + iy) = xy/(x^2 + y^2)$ if $x + iy \neq 0$, and set $f(0) = 0$. Show that $f = u + iv$ is not complex-differentiable at the origin, but that u and v *do* satisfy the Cauchy-Riemann equations there.

The exercise at the right shows that answer is *No*.

In undergraduate complex analysis, it's usual at this point to prove the following partial converse to Proposition 1.2:

Theorem 1.3. *Suppose $f = u + iv$ is defined in an open neighborhood of z_0 in which the partial derivatives of u and v with respect to x and y exist and are continuous. If, in addition, the Cauchy-Riemann equations between u and v hold at z_0, then f is complex differentiable there.*

Useful as it is, Theorem 1.3 masks the "cosmic truth" of the Cauchy-Riemann equations. To see what's really going on we need to realize that a subset Ω of the complex plane can also be regarded as a subset of the euclidean plane $\mathbb{R}^2$, and that a complex-valued function f on $\Omega \subset \mathbb{C}$ is also an $\mathbb{R}^2$-valued function on $\Omega \subset \mathbb{R}^2$. Each of these points of view has its own concept of differentiation, and we'll see shortly that the two are connected by the Cauchy-Riemann equations.

2 *"Real" differentiation in $\mathbb{R}^2$*

Consider an $\mathbb{R}^2$-valued function f defined on an open set $\Omega \subset \mathbb{R}^2$. Fix (in your mind) a point $z_0 \in \mathbb{R}^2$.

Definition 2.1. *To say f is* differentiable at z_0 *means that there is a linear transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ for which*

$$\lim_{|h|\to 0} \frac{|f(z_0+h) - f(z_0) - Ah|}{|h|} = 0\,, \tag{3}$$

Here the "absolute values" denote the "euclidean norm," i.e., for $(x,y) \in \mathbb{R}^2$: $|(x,y)| = \sqrt{x^2+y^2}$.

That the linear transformation A is unique follows from the fact that for any other linear transformation B of $\mathbb{R}^2$ that satisfies (3):

$$\lim_{h\to 0} \frac{|(A-B)h|}{|h|} = 0 \tag{4}$$

In other words, $Ae = Be$ for any unit vector $e \in \mathbb{R}^2$.

Conclusion: $A = B$.

This uniqueness allows us to call the linear transformation A in (3) "the derivative of f at z_0, and write it as $Df(z_0)$.

Let $e_1 = (1,0)$, the unit vector in $\mathbb{R}^2$ along the "x-axis". Upon setting $h - te_1$ in (3) we see that the partial derivative

$$f_x(z_0) = \lim_{t\to 0} \frac{f(x_0+t, y_0) - f(x_0, y_0)}{t} = Df(z_0)e_1 \tag{5}$$

exists. The same argument, now with $h = te_2$ where e_2 is the unit vector "along the y-axis," shows that the partial derivative $f_y(z_0)$ exists, and equals $df(z_0)e_2$. By the linearity of $Df(z_0)$ on $\mathbb{R}^2$, we see that

$$Df(z_0)h = f_x(z_0)h_1 + f_y(z_0)h_2 \qquad (h = (h_1,h_2) \in \mathbb{R}^2). \tag{6}$$

In summary:

Theorem 2.2. *For a point $z_0 \in \mathbb{R}^2$: suppose U is a neighborhood of z_0 and that f is an $\mathbb{R}^2$-valued function on U. If f is differentiable at z_0 in the sense of* (3)*, then:*

(a) The linear transformation A is unique; henceforth we call it $Df(z_0)$.

(b) The partial derivatives f_x and f_y (cf. (5)*) both exist at z_0, and equal $Df(x_0)e_1$ and $Df(x_0)e_2$ respectively, hence*

(c) $Df(z_0)h$ *is given by* (6) *for each* $h = (h_1, h_2) \in \mathbb{R}^2$.

What's missing here is a useful sufficient condition for real differentiability. Here is one:

Theorem 2.3. *Suppose f is an R^2-valued function defined in an open subset Ω of $\mathbb{R}^2$. If f is "smooth" on Ω (in the sense that the partial derivatives of its real and imaginary parts, with respect to x and y both exist and are continuous on there) then f is real-differentiable at every point of Ω.*

Proof. See [4, Thm 9.21, page 218], for example. □

Note that in this case the real derivative Df is continuous on Ω, and so actually have an iff statement:

> *$f\colon \Omega \to \mathbb{R}^2$ is real-differentiable on Ω, with Df continuous there, if and only if f is smooth on Ω.*

2.1 *Complex differentiation II*

Let's rewrite our original definition of "complex derivative" (Definition 1.1) in the language of Definition 2.1. As usual, we begin with a complex-valued function f defined on a neighborhood of the point $z_0 \in \mathbb{C}$.

Definition 2.4. *To say f is* (complex) differentiable at z_0 *means that there is a complex number $a := f'(z_0)$ such that*

$$\lim_{h\to 0} \frac{|f(z_0+h) - f(z_0) - ah|}{|h|} = 0 \tag{7}$$

Lurking within this definition is the (complex) linear transformation $A\colon \mathbb{C} \to \mathbb{C}$ defined for $h \in \mathbb{C}$ by $A(h) = ah$. In fact, every linear transformation on $\mathbb{C}$ has this form (with $a = A(1)$). Thus our original definition of complex differentiability is the same as that of real differentiability, with (real) linearity of the derivative replaced by complex linearity.

Suppose, then that the complex-valued function f defined on a neighborhood Ω of a point $z_0 \in \mathbb{C}$ is real-differentiable at z_0. When is it complex-differentiable?

By real differentiability we have at z_0, for each $h = (h_1, h_2) \in \mathbb{R}^2$:

$$\begin{aligned}(Df)h = f_x h_1 + f_y h_2 &= f_x\left(\frac{h+\overline{h}}{2}\right) + f_y\left(\frac{h-\overline{h}}{2i}\right)\\ &= \frac{f_x - if_y}{2}\,h + \frac{f_x + if_y}{2}\,\overline{h}\end{aligned}$$

Giving in to temptation, let's define

$$f_z := \frac{f_x - if_y}{2} \quad \text{and} \quad f_{\overline{z}} := \frac{f_x + if_y}{2} \tag{8}$$

so

$$(Df)h = f_z\,h + f_{\overline{z}}\,\overline{h} \tag{9}$$

which makes it clear that the real-differentiable f is *complex differentiable* at z_0 if and only if $f_{\bar{z}}(z_0) = 0$, in which case $f_z(z_0)$ is the complex derivative at z_0.

If we write df for Df, dz for h and $d\bar{z}$ for $\bar{h}$, and use partial-derivative notation instead of subscripts, then (9) assumes the evocative form

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}$$

.

What does $f_{\bar{z}}(z_0) = 0$ mean? Writing $f = u + iv$ (real and imaginary parts) we obtain, at z_0:

$$2f_{\bar{z}} = f_x + if_y = (u_x + iv_x) + i(u_y + iv_y) = (u_x - v_y) + i(v_x + u_y)\,,$$

so $f_{\bar{z}} = 0$ iff $u_x = v_y$ and $u_y = -v_x$, i.e, iff the Cauchy-Riemann equations (2) hold.

In summary:

Theorem 2.5. *For a complex-valued function f defined in a neighborhood of a point $z_0 \in \mathbb{C}$, the following are equivalent:*

(a) f is complex differentiable at z_0.

(b) f is real differentiable at z_0 and the linear transformation $Df(z_0)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is complex-linear.

(c) f is real differentiable at z_0 and $f_{\bar{z}}(z_0) = 0$.

(d) f is real differentiable at z_0, and the Cauchy-Riemann equations relating its real and imaginary parts hold at z_0.

3 *Cauchy-Riemann and Analyticity*

Suppose f is a complex-valued function f defined on an open subset Ω of $\mathbb{C}$

Definition 3.1. *To say f is* analytic *on Ω means that it's complex-differentiable at every point of Ω.*

Sometimes people say that a complex-valued function is *analytic at a point* $z_0 \in \mathbb{C}$. This means that the function is defined and complex-differentiable *in a neighborhood of* z_0.

Motivated by equations (8) and (9) above, we define the differential operators

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}} := \tfrac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \partial = \frac{\partial}{\partial z} := \tfrac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

whereupon the Cauchy-Riemann equations (2),as condensed in Theorem 2.5(c), can now be written simply: $\bar{\partial} f = 0$. With this notation we can rephrase the content of Theorem 2.5 like this:

Theorem 3.2. *A complex-valued function f defined on an open subset Ω of $\mathbb{C}$ is* analytic *on Ω if and only if: on Ω it is real-differentiable and $\bar{\partial} f \equiv 0$.*

We know from Theorem 2.3 that f is real-differentiable on Ω whenever it is smooth there. Thus:

Corollary 3.3. *Suppose the complex-valued function f is smooth on an open subset Ω of $\mathbb{C}$. Then f is analytic on Ω iff $\bar{\partial} f \equiv 0$ there.*

The operator $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ maps $C^1(\Omega)$ linearly into $C(\Omega)$, obeys the usual properties of differentiation. In particular, at each point of Ω:

(a) $\bar{\partial}1 = 0$ and $\bar{\partial}\bar{z} = 1$

(b) $\bar{\partial}(fg) = f\,\bar{\partial}g + (\bar{\partial}f)g$ for each pair f, g of functions in $C^1(\Omega)$.

(c) $\bar{\partial}(fg) = f\,\bar{\partial}g$ if $f, g \in C^1(\Omega)$ and f is analytic on Ω.

(d) $\bar{\partial}\bar{z}^n = n\bar{z}^{n-1}$ for $n = 1, 2, \ldots$.

The proofs (a) and (b) follow from the definition of $\bar{\partial}$ and the usual properties of differentiation. (c) follows from (b) and $\bar{\partial}f = 0$, and (d) from (a) and (b) by induction.

In particular, given a polynomial in the real variables x and y, with complex coefficients, these properties tell us that the polynomial is analytic on $\mathbb{C}$ precisely when, in its representation as a linear combination of terms $z^j\,\bar{z}^j$, terms involving $\bar{z}$ do not occur.

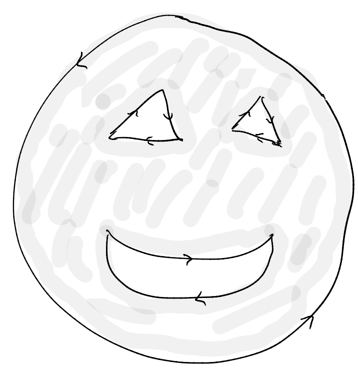
4 *The Fork-in-the-Road*

In this section we'll derive a remarkable theorem that splits complex analysis into two branches: the usual classical one based on the Cauchy integral representation of analytic functions, and a second, seemingly "orthogonal" branch based on the "nonhomogeneous Cauchy-Riemann equation:" $\bar{\partial}u = f$. In subsequent sections we'll see how these two seemingly disparate branches coalesce into a fascinating hybrid field of study.

The key to everything is a two-variable version of the Fundamental Theorem of Integral Calculus, published in 1828 by the British mathematician George Green, and known to every student of second-year calculus.

4.1 *Green's Theorem*

Definition 4.1. *By a "Green domain" we'll mean an open subset of $\mathbb{R}^2$ bounded by a finite, pairwise-disjoint collection of simple, piecewise smooth curves.*



A "Green domain", with its positively oriented boundary

For a Green set Ω we'll denote by $\partial\Omega$ the boundary of Ω, oriented positively (i.e., such that when you traverse the boundary in the direction of orientation, your left hand is in Ω).

Theorem 4.2 (Green's Theorem)**.** *Suppose Ω is a Green domain, and that P and Q are complex-valued functions that are smooth on a neighborhood of the closure of Ω. Then*

$$\int_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,.$$

Green's Theorem is usually stated for real-valued functions P and Q, however the complex-valued extension is easily obtained by applying the real version separately to real and imaginary parts.

The $\bar{\partial}$ operator provides a particularly useful special case of Green's Theorem:

Corollary 4.3. *For Ω as above and f a complex-valued function that is smooth on a neighborhood of the closure of Ω:*

$$\int_{\partial\Omega} f(z)\,dz = 2i \int_{\Omega} \bar{\partial} f\,dA$$

where dA denotes the usual area measure on Ω.

Proof. Upon noting that $dz = dx + i\,dy$, we have from Green's Theorem:

$$\int_{\partial\Omega} f(z)\,dz = \int_{\partial\Omega} f\,dx + if\,dy = \int_{\Omega} \left(\frac{\partial(if)}{dx} - \frac{\partial f}{dy}\right) dA$$
$$= i \int_{\Omega} \left(\frac{\partial f}{dx} + i\frac{\partial f}{dy}\right) dA = 2i \int_{\Omega} \frac{\partial f}{\partial \bar{z}}\,dA\,. \qquad \square$$

The following consequence of Green's Theorem has profound implications for the both the classical and the modern study of analytic functions.

Corollary 4.4 (The "Fork-in-the-Road" Theorem)**.** *Suppose Ω is a Green domain and f is a complex-valued function that is smooth on a neighborhood of the closure of Ω. Then for every point $z_0 \in \Omega$:*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz - \frac{1}{\pi} \int_{\Omega} \frac{(\bar{\partial} f)(z)}{z - z_0}\,dA(z) \tag{10}$$

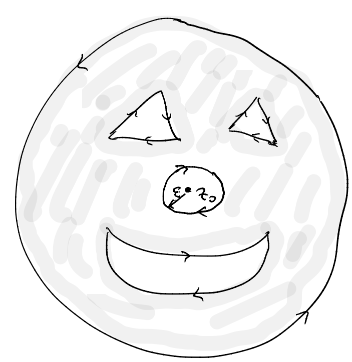
Why the "fork in the road"? If f is analytic in Ω then $\bar{\partial} f \equiv 0$ there, so (10) reduces to the classical Cauchy integral formula:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - z_0}\,dz \qquad (z_0 \in \Omega). \tag{11}$$

If, on the other hand, $f \in C^1(\Omega)$ has *compact support*, then, since $f \equiv 0$ on $\partial\Omega$, (10) reduces to *Pompieu's formula*

$$f(z_0) = -\frac{1}{\pi} \int_{\Omega} \frac{(\bar{\partial} f)(z)}{z - z_0}\,dA(z) \qquad (z_0 \in \Omega). \tag{12}$$

From the Cauchy formula follows the classical theory of analytic functions, while the Pompieu formula seems to lead in an entirely different "real-variable" direction. We'll explore this direction in the next section, after which we'll see that, quite surprisingly, the two streams of inquiry merge into a remarkable synthesis of real and complex methods.



The Green domain Ω_ε

Proof of Corollary 4.4. Fix (for the rest of the argument) $z_0 \in \Omega$ and (temporarily) $\varepsilon > 0$, chosen small enough that the z_0-centered closed disc Δ_ε of radius ε lies entirely in Ω. Let $\Omega_\varepsilon = \Omega \backslash \Delta_\varepsilon$. This is a Green domain whose boundary consists of the union of $\partial\Omega$ and the boundary γ_ε of Δ_ε, oriented in the clockwise direction. Thus $\partial\Omega_\varepsilon$, the boundary of Ω_ε has positive orientation. Upon applying Corollary 4.3 with $f(z)$ replaced by $f(z)/(z - z_0)$, we obtain:

$$\int_{\partial\Omega_\varepsilon} \frac{f(z)}{z - z_0}\,dz = 2i \int_{\Omega_\varepsilon} \bar{\partial}\left(\frac{f(z)}{z - z_0}\right) dA(z) = 2i \int_{\Omega_\varepsilon} \frac{(\bar{\partial} f)(z)}{z - z_0}\,dA(z), \tag{13}$$

where the last equality reflects the fact that, in the integrand of the right-hand integral the factor $1/(z-z_0)$ is analytic on Ω_ε, and therefore "treated as a constant" by the $\bar\partial$ operator.

The left-hand integral (13) is the sum of integrals over $\partial\Omega$ and γ_ε. For the latter of these we use the parameterization $z = z_0 + \varepsilon e^{i\theta}$ with $0 \le \theta < 2\pi$. Thus

$$\begin{aligned}
\mathrm{LHS}(13) &= \int_{\partial\Omega} + \int_{\gamma_\varepsilon} \frac{f(z)}{z-z_0}\,dz \\
&= \int_{\partial\Omega} \frac{f(z)}{z-z_0}\,dz - \int_0^{2\pi} \frac{f(z_0+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \underbrace{i\,\varepsilon\, e^{i\theta}\,d\theta}_{dz} \\
&= \int_{\partial\Omega} \frac{f(z)}{z-z_0}\,dz - i\int_0^{2\pi} f(z_0+\varepsilon\, e^{i\theta})\,d\theta
\end{aligned}$$

By the continuity of f at z_0, the second integral in the last line above $\to 2\pi f(z_0)$ as $\varepsilon \to 0$. Thus

$$\lim_{\varepsilon\to 0+} \int_{\partial\Omega_\varepsilon} \frac{f(z)}{z-z_0}\,dz = \int_{\partial\Omega} \frac{f(z)}{z-z_0}\,dz - 2\pi\, i\, f(z_0). \tag{14}$$

Now look at the right-hand side of (13), and note that the measure $|z-z_0|^{-1}dA(z)$ gives finite mass to any measurable bounded subset of the plane (hint: change to polar coordinates). This, along with the fact that $\bar\partial f$ is bounded on Ω (because it is continuous on a neighborhood of the closure of Ω), shows that

$$\lim_{\varepsilon\to 0+} \int_{\Omega_\varepsilon} \frac{(\bar\partial f)(z)}{z-z_0}\,dA(z) = \int_{\Omega} \frac{(\bar\partial f)(z)}{z-z_0}\,dA(z)\,.$$

This, along with (13) and (14), proves (10). □

5 *The non-homogeneous Cauchy-Riemann Equation*

We've seen that (at least for smooth functions on open subsets of the plane) the "homogeneous" Cauchy-Riemann equation $\bar\partial f = 0$ characterizes analyticity. In this section we introduce the "nonhomogeneous" Cauchy-Riemann equation $\bar\partial u = f$. In the following sections we show that it, too, has important connections with analyticity. As usual, Ω always denotes an open (non-empty) subset of the plane.

Theorem 5.1. *For each smooth function $f\colon \Omega \to \mathbb{C}$, there exists a function $u\colon \Omega \to \mathbb{C}$ such that $\bar\partial u = f$ on Ω.*

Go back to $\Omega = \mathbb{C}$ in the statement?

Initially we'll prove, and apply, this result only for the special case $\Omega = \mathbb{C}$, postponing the proof of the full theorem to a yet-to-be-written section.

Proof (for $\Omega = \mathbb{C}$). We break the argument into three parts.

STEP I: SUPPOSE f HAS COMPACT SUPPORT. In this case the function u we seek is defined by the integral:

"$\int$" means "integral over $\mathbb{R}^2$." The one in (15) exists because the integrand has compact support, and the measure $|\zeta - z|^{-1}\, dA(\zeta)$ is finite on compact sets.

$$u(z) = \int \frac{f(\zeta)}{\zeta - z}\, dA(\zeta) \qquad (z \in \mathbb{C}). \tag{15}$$

The existence of the integral, and the fact that u smooth on $\mathbb{C}$ both follow from the finiteness of the measure $|\zeta - z|^{-1}\, dA(\zeta)$ the fact that f belongs to this class, and has compact support. For each z we may replace ζ by $\zeta + z$ in the integrand, obtaining:

$$u(z) = \int \frac{f(\zeta + z)}{\zeta}\, dA(\zeta).$$

Now "unfix" z, differentiate both sides of this last expression, and interchange the order of integration and differentiation:

$$\begin{aligned}
(\bar\partial u)(z) &= \bar\partial \int \frac{f(\zeta+z)}{\zeta}\, dA(\zeta) = \int \bar\partial_z \frac{f(\zeta+z)}{\zeta}\, dA(\zeta) \\
&= \int \frac{(\bar\partial_z f)(\zeta+z)}{\zeta}\, dA(\zeta) = \int \frac{(\bar\partial_\zeta) f(\zeta+z)}{\zeta}\, dA(\zeta) \\
&= \int \frac{(\bar\partial f)(\zeta)}{\zeta - z}\, dA(\zeta) \\
&= f(z)
\end{aligned}$$

where the last equality follows from (12). Thus $\bar\partial u = f$ on $\mathbb{C}$, as desired.

The rest of the proof consists of showing that an arbitrary $f\colon \mathbb{C} \to \mathbb{C}$ that is smooth on $\mathbb{R}^2$ can be decomposed into a convergent series of smooth functions that have compact support (Step II). The first part of the argument can then be applied to each piece of the series, after which (in Step III) the results are adjusted to provide a suitably convergent series of solutions.

STEP II: A SMOOTH "PARTITION OF UNITY." As a first step in making the desired decomposition of a function in $C^1(\mathbb{C})$, let's carry out a similar construction on the real line for the function $f \equiv 1$. The naive decoposition $\psi_n = 1$ on $[n, n+1]$ and 0 otherwise, does indeed result in $\sum_{n\in\mathbb{Z}} f_n \equiv 1$, but with the ψ_n's no longer in $C^1(\mathbb{R})$. However some judicious smoothing removes this objection. Almost anything will work, but let's be specific.

For $n \in \mathbb{Z}$ choose $\psi_n \in C^1(\mathbb{R})$ with $\psi_n(x) \geq 1$ for $x \in [n, n+1]$, and $= 0$ for x outside the interval $(n-1, n+2)$. Since each $x \in \mathbb{R}$ belongs to at most two of the intervals $(n-1, n+2)$ we see that $\psi_n(x)$ is non-zero for at most three values of n, hence the infinite series $\sum_{n\in\mathbb{Z}} \psi_n$ converges pointwise on $\mathbb{R}$ to a function in $C^1(\mathbb{R})$ whose value is positive at every point. Now let

$$\varphi_n(x) = \frac{\psi_n(x)}{\sum_k \psi_k(x)} \qquad (x \in \mathbb{R}, n \in \mathbb{Z}). \tag{16}$$

Then each φ_n is in $C^1(\mathbb{R})$, at every point of $\mathbb{R}$ is ≥ 0, with values > 0 on $[n, n+1]$. Furthermore, each φ_n has support contained in the open interval $I_n = (n-1, n+2)$; and finally: the series $\sum_n \varphi_n(x)$ converges, and $= 1$, at every point of $\mathbb{R}$.

We call the collection $\{\varphi_n : n \in \mathbb{Z}\}$ a *partition of unity subordinate to the open cover* $\{I_n\}$ *of* $\mathbb{R}$. We can now decompose any $f \in C^1(\mathbb{R})$ as a convergent sum $f = \sum_n f_n$ of C^1 functions, by writing $f_n := \varphi_n f$ for each $n \in \mathbb{Z}$.

Our real goal, however, is to do something like this for arbitrary $f \in C^1(\mathbb{C})$. For this, begin with $\{\varphi_n\}$ as above, and define for $n = 0, 1, 2, \ldots$

$$\Phi_n(z) = \varphi_n(|z|) \qquad (z \in \mathbb{C}).$$

Set $A_n = \{z \in \mathbb{C} : n-1 < |z| < n+1\}$ if $n = 1, 2, \ldots$, and let $A_0 = \{0 \leq |z| < 2\}$. Then the collection of open sets $\{A_n\}_{n=1}^{\infty}$ covers $\mathbb{C}$, and Then the collection of functions $\{\Phi_n\}_0^{\infty}$ is a C^1 partition of unity on $\mathbb{R}^2$ subordinate to this open cover.

Thus, given $f \in C^1(\mathbb{C})$ and a non-negative integer n, let $f_n = \Phi_n f$. Then $f = \sum_n f_n$, with each $f_n \in C^1(\mathbb{C})$ and having compact support. It follows that for each n the non-homogeneous Cauchy-Riemann equation $\bar{\partial} u_n = f_n$ has, by Step I, a C^1 solution on $\mathbb{C}$.

Step III: Completing the proof. Continuing the work of the paragraph above, the problem now is that the series $\sum_n u_n$—with u_n the solution to $\bar{\partial} u_n = f_n$ obtained by Step I—need not converge. However, for $n \geq 2$ we have $f_n(z) = 0$ for each point z in a neighborhood of the closed disc $|z| \leq n-1$. Thus u_n is analytic in that neighborhood, and is the sum of its Maclaurin series, uniformly convergent for $|z| \leq n-1$. Consequently me may choose a partial sum p_n of this series with $|u_n(z) - p_n(z)| < 2^{-n}$ for $|z| \leq n-1$. Let $v_n = u_n - p_n$, a function analytic in a neighborhood of the disc $\{|z| \leq n-1\}$. It follows from the Weierstrass M Test the sum $\sum_{k=n}^{\infty} v_k$, each term of which is analytic in a neighborhood of the disc $\{|z| \leq n-1\}$ converges uniformly and absolutely on that disc, its sum therefore being analytic there, with its complex derivative represented in the same way by the series of derivatives of the v_k.

Conclusion. The series $\sum_{n=0}^{\infty} v_n$ converges uniformly on compact subsets of the plane to a C^1 function u that satisfies the non-homogeneous Cauchy-Riemann equation $\bar{\partial} u = f$ on $\mathbb{C}$.

6 *Poles and Zeros*

Fix, for a while, a point $z_0 \in \mathbb{C}$. If U is a neighborhood of z_0,l then we call $U \backslash \{z_0\}$ a "punctured" neighborhood of z_0. If a function f is analytic in a punctured neighborhood of z_0, then there are only three possibilities:

(a) f is bounded in a (possibly smaller) punctured neighborhood of z_0. In this case f extends analytically to all of U, and z_0 is called a "removeable singularity" of f.

(b) There exists a positive integer n such that $\lim_{z \to z_0} |z - z_0|^n$ exists and is not zero. In this case z_0 is called a "pole" of f of "order n."

(c) z_0 is neither a removeable singularity nor a pole. In this case f maps every punctured neighborhood of z_0 onto a dense subset of the plane (the Casorati-Weierstrass Theorem), in fact, onto the whole plane minus at most one point, with each image value being assumed infinitely often (Picard's Great Theorem).

Example: $f(z) = e^{1/z}$, with $z_0 = 0$.

The origin of this trichotomy is the following result, which follows from the Cauchy Integral Theorem (Theorem 11).

Theorem. *If f is analytic in a punctured neighborhood of z_0, then there exists a doubly-infinite sequence $(a_n : n \in \mathbb{Z})$ such that*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n \tag{17}$$

where the series on the right converges in the largest punctured neighborhood in which f is analytic.

The series on the right is called a *Laurent series*. The tripartite classification listed above for the possible singularities of f at z_0 corresponds to the exhaustive list below of the possibilities for the sequence (a_n) of coefficients for negative values of the index n:

(a) If $a_n = 0$ for all $n < 0$ then z_0 is a *removable* singularity.

(b) If $a_n \neq 0$ for some, but only finitely many $n < 0$, then z_0 is a *pole* of f.

(c) Otherwise $a_n \neq 0$ for infinitely many $n < \infty$, and z_0 is an *essential* singularity.

In all cases, the series $\sum_{n<0} a_n (z - z_0)^n$ is called the *principal part* of f at z_0.

Theorem 6.1 (G. Mittag-Leffler, 1876). *For a sequence (z_n) of distinct points in the complex plane with $|z_n| \to \infty$, and a sequence (m_n) of positive integers, there exists a function f meromorphic on $\mathbb{C}$, analytic except at each point z_n where it has a pole of order m_n.*

For primary references to this result, and an exhaustive history, see [8].

Proof. We first create a "smooth prototype" solution to our "Mittag-Leffler Problem." Choose open discs $\Delta_n = \{|z - z_n < r_n\}$, with radii chosen so that these discs are pairwise disjoint. For each n choose a "bump function" $b_n \in C^1(\mathbb{R}^2)$ that takes value $\equiv 1$ on a neighborhood of z_n, and with compact support contained in Δ_n. Let $f_n(z) = (z - z_n)^{-m_n}$ and set $\varphi = \sum_n \varphi_n f_n$, noting that at each point of the plane at most one term of the sum is non-zero. Thus φ belongs to $C^1(\mathbb{R}^2\backslash\{z_n\})$, and for each n is equal to f_n in a neighborhood of z_n.

Our goal is to "correct" φ so that the new function is analytic on $\mathbb{C}\backslash\{z_n\}$, but without losing the pole that φ has at each z_n. In other words, we seek a function $u \in C^1(\mathbb{C})$ that is analytic in a neighborhood of each z_n, and for which $f := \varphi - u$ is analytic on $\mathbb{C}\backslash\{z_n\}$.

To this end, note that $\bar{\partial}\varphi$ is C^1 on $\mathbb{R}^2\backslash\{z_n\}$, and is analytic in a "punctured" neighborhood of z_n (i.e., in a neighborhood of z_n, with z_n removed). Thus, the function g defined on $\mathbb{C}$ by $g = \bar{\partial}\varphi$ off $\{z_n\}$, and $= 0$ at each z_n, is C^1 on all of $\mathbb{C}$, and, for each n, vanishes identically on a neighborhood of z_n. By Theorem 5.1 there exists $u \in C^1(\mathbb{C})$ such that $\bar{\partial}u = g$ at each point of $\mathbb{C}$. In particular, upon setting $f = \varphi - u$ we have

$$\bar{\partial}f \;=\; \bar{\partial}\varphi - \bar{\partial}u \;=\; \bar{\partial}\varphi - g \;=\; 0 \quad \text{on} \quad \mathbb{C}\backslash\{z_n\}.$$

Thus f is analytic on $\mathbb{C}\backslash\{z_n\}$. Furthermore, for each n we have, in a neighborhood of z_n: $\varphi = f_n$ and $g = 0$ (so $\bar{\partial}u = 0$, so u is analytic in this neighborhood).

Conclusion: f is analytic on $\mathbb{C}\backslash\{z_n\}$, and for each n is equal to

$$f_n \;+\; \text{a function analytic on a neighborhood of } z_n,$$

i.e., f has a pole of order m_n at z_n. □

Upon replacing the function f created in Theorem 6.1 by $1/f$, we obtain an important result about zeros (originally due to Weierstrass).

Corollary 6.2. *Given $\{z_n\}$ and $\{m_n\}$ as in Theorem 6.1: There exists an entire function f that has for each n, a zero of order m_n at z_n, and has no other other zeros.*

Remark. In the proof given here for Theorem 6.1, each function f_n could equally well be chosen to be analytic in a punctured neighborhood of z_n, with either a pole or an essential singularity at z_n. There would then result a function f analytic on $\mathbb{C}\backslash\{z_n\}$ with "principal part" at z_n prescribed by the function f_n.

7 *The ring of entire functions*

We'll use $\mathscr{E}$ to denote the collection of entire functions, i.e., those functions that are analytic on the "entire" complex plane. When endowed with the operations of pointwise addition and multiplication, $\mathscr{E}$ becomes a *commutative ring*.

7.1 *Ideals in $\mathscr{E}$*

Definition 7.1. An *ideal* in $\mathscr{E}$ is a subring $\mathscr{I}$ that is "super-multiplicatively-closed" in the sense that for each $g \in \mathscr{I}$ the product fg lies in $\mathscr{I}$ for every $f \in \mathscr{I}$.

If we were talking about general, not-necessarily-commutative rings, such $\mathscr{I}$ would be a *left* ideal.

For the ring $\mathscr{E}$, and indeed for any ring with a multiplicative unit 1, an ideal is equal to the whole ring if and only if it contains 1.

Examples (of ideals in $\mathscr{E}$).

(a) $\mathscr{I} = \{f \in \mathscr{E} \colon f(0) = 0\}$.

(b) $\mathscr{I} =$ all $f \in \mathscr{E}$ that vanish on a given subset of $\mathbb{C}$. E.g., for $n \in \mathbb{N}$:

$$\mathscr{I}_n = \{f \in \mathscr{E} \colon f(k) = 0 \; \forall \, k \geq n\}.$$

(c) A proper ideal that has no common zero: $\mathscr{I} = \bigcup_{n \in \mathbb{N}} \mathscr{I}_n$.

The ideal *generated by* a set $\mathscr{S}$ of elements in $\mathscr{E}$ is just the smallest ideal containing $\mathscr{S}$; notation $\langle \mathscr{S} \rangle$. It's easy to check that $\langle \mathscr{S} \rangle$ is just the set of all sums of finitely many terms of the form sf, where $s \in \mathscr{S}$ and $f \in \mathscr{E}$. In particular, if $f_1, f_2, \ldots, f_n \in \mathscr{E}$, then

$$\langle f_1, f_2, \ldots f_n \rangle = \left\{ \sum_{j=1}^{n} f_j \, g_j \colon g_1, g_2, \ldots, g_n \in \mathscr{E} \right\}.$$

Note that the (possibly empty) collection of common zeros of an ideal is just the collection of common zeros of its set of generators (there always is such a set: every ideal is generated by itself).

The examples listed above suggest the following

Question. *Does every finitely generated* proper *ideal of $\mathscr{E}$ have a common zero*?

Equivalently:

> *Must every proper ideal of $\mathscr{E}$ without a common zero be infinitely generated?*

Exercise. Prove that Example (c) above is infinitely generated.

Our $\overline{\partial}$ point of view provides an answer; we present this for the special case of doubly generated ideals.

7.2 *Doubly generated ideals in $\mathscr{E}$*

Theorem 7.2. *Suppose f_1 and f_2 are entire functions with no common zero. Then $\langle f_1, f_2 \rangle = \mathscr{E}$.*

Proof. It's enough to prove that $1 \in \langle f_1, f_2 \rangle$, i.e., that there exist $g_1, g_2 \in \mathscr{E}$ such that

$$f_1 \, g_1 + f_2 \, g_2 = 1 \,. \tag{18}$$

To this end we first form "smooth prototypes" for the g's, which we'll then seek to "correct" using the non-homogeneous Cauchy-Riemann equation. Since f_1 and f_2 have no common zero, the smooth function $|f_1|^2 + |f_2|^2$ is > 0 at every point of the plane, so the functions

$$\gamma_j := \frac{\overline{f_j}}{|f_1|^2 + |f_2|^2} \qquad (j = 1, 2)$$

are smooth on $\mathbb{C}$, and

$$f_1\gamma_1 + f_2\gamma_2 = 1 \quad \text{on } \mathbb{C}. \tag{19}$$

We seek to "correct" the γ's without losing (19).

Claim. There exists a smooth function u on $\mathbb{C}$ such that

$$g_1 := \gamma_1 + u f_2 \quad \text{and} \quad g_2 := \gamma_2 - u f_1 \tag{20}$$

are entire functions.

We'll be done once we prove the Claim, since (18) will then follow from (20) and (19).

Proof of the Claim.[1]

Upon applying the $\bar\partial$ operator to both sides of each equation in (20), and using the fact that "$\bar\partial$ thinks entire functions are constants," we see that the smooth function we seek must be chosen so that

$$0 = \bar\partial\gamma_1 + (\bar\partial u)f_2 \quad \text{and} \quad 0 = \bar\partial\gamma_2 - (\bar\partial u)f_1 \tag{21}$$

Multiply the first of these equations by γ_2, the second by γ_1, subtract, and use (18). *Conclusion*: If u satisfies both equations of (21), then

$$\bar\partial u = \gamma_1\,\bar\partial\gamma_2 - \gamma_2\,\bar\partial\gamma_1 \tag{22}$$

Since the right-hand side of this last equation is smooth on $\mathbb{C}$, we can, by Theorem 5.1, find a smooth solution u.

It remains to check that any function u that satisfies (22) satisfies both equations of (21). We'll do this for the first equation of (21). On the right-hand side of this equation, substitute the expression for $\bar\partial u$ from (22). We find that

$$\bar\partial\gamma_1 + (\bar\partial u)f_2 = \bar\partial\gamma_1 + (\gamma_1\bar\partial\gamma_2 - \gamma_2\bar\partial\gamma_1)f_2 \tag{23}$$

Upon applying the $\bar\partial$ operator to both sides of (19) we find that $(\bar\partial\gamma_2)f_2 = -(\bar\partial\gamma_1 f_1)$, which, when substituted on the right-hand side of (23) yields

$$\begin{aligned}
\bar\partial\gamma_1 + (\bar\partial u)f_2 &= \bar\partial\gamma_1 - (\bar\partial\gamma_1)\gamma_1 f_1 - (\bar\partial\gamma_1)\gamma_2 f_2 \\
&= \bar\partial\gamma_1 - \bar\partial\gamma_1(\gamma_1 f_1 + \gamma_2 f_2) \\
&= \bar\partial\gamma_1 - \bar\partial\gamma_1 \\
&= 0.
\end{aligned}$$

Thus u satisfies the first equation of (21). The verification that it satisfies the second one is similar (alternatively, the argument of sidenote 2 above shows that these equations are equivalent, in that any solution of one is a solution of the other).

This proves the Claim, and with it, the Theorem. □

[1] It may seem daunting that we need to find *one* function u to fulfill *both* equations of (20). But that is not really the problem; a little bit of algebra based on (19) shows that if u satisfies the first equation of (20), and g_2 is defined by the second one, then the pair g_1, g_2 satisfies (18).

An argument that's perhaps more transparent starts by applying the $\bar\partial$ operator to both sides of the first equation of (20), and solving for $\bar\partial u$. Then the equation to be solved for u becomes:

$$\bar\partial u = -(\bar\partial\gamma_1)/f. \tag{*}$$

A straightforward calculation, using only the definition of γ_1 and the fact that $\bar\partial$ obeys all the usual rules of differentiation, shows that

$$\bar\partial\gamma_1 = f_2\frac{f_2\overline{\partial f_1} - f_1\overline{\partial f_2}}{f_1\overline{f_1} + f_2\overline{f_2}}$$

so $(\bar\partial\gamma_1)/f_2$ is, in fact, C^1 on the whole plane, hence we can find a C^1 solution to the $\bar\partial$ equation (*).

7.3 Zeros

To move further in our study of ideals in the ring of entire functions, we need to be precise about the "set of zeros" of an analytic function.

Recall the Uniqueness Theorem which asserts that if a function analytic function on an open set Ω assumes the value zero on a subset having a limit point in Ω, then that function must vanish identically on Ω. In particular, a nontrivial analytic function cannot assume the value zero on an uncountable set.

As is the case for polynomials, the zeros of analytic functions come in "integer strengths." Every zero z_0 of f has an *order* $n = n(f, z_0)$, a positive integer such that $\lim_{z \to z_0} \frac{f(z)}{(z-z_0)^n}$ exists (in $\mathbb{C}$) and is not zero. When referring to the zero-set of an analytic function it is important to keep track of their orders. Thus, instead of a zero-set for f analytic on Ω, we'll deal with a *zero list*. This will be an ordered list (either empty, finite, or countably infinite) $Z(f) := (z_n)$ consisting of the zeros of f arranged in order of increasing absolute values, and for which each zero is repeated according to its multiplicity.

Thus f has a zero of order n at z_0 if and only if $1/f$ has a pole of order n there.

Example. If $f(z) = (z-1)(z+2)^2(z-i)^3$, then $Z(f) = (1, i, i, i, 2, 2)$ or $(i, i, i, 1, 2, 2)$ (if different zeros of the same modulus can occur in any order.

If $\mathscr{S}$ is a *set* of analytic functions, then $Z(\mathscr{S})$ will denote its *list of common zeros*, meaning that each zero z_0 common to all the functions in $\mathscr{S}$ will occur $n = \min_{f \in \mathscr{S}} n(f, z_0)$ times. Note that:

Example, the common-zero list of polynomials

$$\{z(z-1)(z-2)^2, z^2(z-2)^3(z-3)\}$$

is $(0, 2, 2)$.

(a) If $Z(\mathscr{S})$ is not empty then, by Corollary 6.2 there will be an entire function f with $Z(f) = Z(\mathscr{S})$, in which case $\mathscr{S}/f$ will have no common zero (i.e., $Z(\mathscr{S}/f)$ will be empty).

(b) $Z(f) \subset Z(g) \iff \langle g \rangle \subset \langle f \rangle$. [2]

[2] *Proof.* $Z(f) \subset Z(g) \iff \frac{g}{f} \in \mathscr{E}$ $\iff g = fe$ $\exists$ entire function e $\iff g \in \langle f \rangle \iff \langle g \rangle \subset \langle f \rangle$. $\square$

In case $\mathscr{S}$ consists of just two functions, observation (a) yields the following generalization of Theorem 7.2:

Corollary 7.3. *Every doubly generated ideal of $\mathscr{E}$ is singly generated.*

Proof. Fix f_1 and f_2 in $\mathscr{E}$. If these functions have no common zero then, by Theorem 7.2, they generate $\mathscr{E}$, which is also generated by the constant function 1. So suppose there is a common zero, in which case $Z := Z(\{f_1, f_2\})$ is not empty. Choose $f \in \mathscr{E}$ with $Z(f) = Z$. Then the set $\{f_1/f, f_2/f\}$ has no common zero, so the ideal it generates is $\mathscr{E}$, i.e., there exist entire functions g_1 and g_2 such that

$$\frac{f_1}{f} g_1 + \frac{f_2}{f} g_2 = 1 \quad \text{i.e.,} \quad f_1 g_1 + f_2 g_2 = f$$

Thus f lies in $\langle f_1, f_2 \rangle$, so $\langle f \rangle \subset \langle f_1, f_2 \rangle$.

Conversely, suppose $h \in \langle f_2, f_2 \rangle$, i.e., suppose $h = f_1 g_1 + f_2 g_2$ for some entire functions g_1 and g_2. Then $Z = Z(\{f_1, f_2\}) \subset Z(h)$ Since f was chosen so that $Z(f) = Z$ se see from observation (b) above that $h \in \langle f \rangle$. Thus $\langle f_1, f_2 \rangle \subset \langle f \rangle$, so there is equality. $\square$

7.4 *Finitely generated ideals in $\mathscr{E}$*

In ring theory, a singly generated ideal is called *principal*. So far we know that doubly generated ideals in the ring of entire functions are principal. It's now a simple matter to extend this result to all finitely generated ideals

Theorem 7.4. *Every finitely generated ideal in $\mathscr{E}$ is principal.*

Proof. We'll prove a more precise statement:

> Given $f_1, f_2, \ldots f_n \in \mathscr{E}$, and $f \in \mathscr{E}$ with $Z(f) = Z(\{f_1, f_2, \ldots f_n\})$, we have $\langle f_1, f_2, \ldots f_n \rangle = \langle f \rangle$.

We proceed by induction on the number n of generators. If $n = 1$ there is nothing to prove, and we've already settled the case $n = 2$. Assume the result is true for some $n \geq 1$. We desire to establish it for $n + 1$.

To this end, let $\mathscr{I} = \langle f_1, \ldots, f_{n+1} \rangle$. Choose $\varphi \in \mathscr{E}$ with $Z(\varphi) = Z(\{f_1, f_2, \ldots f_n\})$. By the induction hypothesis, $\langle \varphi \rangle = \langle f_1, f_2, \ldots, f_n \rangle$. Now we need only show that $\langle \varphi, f_{n+1} \rangle = \langle f_1, \ldots, f_{n+1} \rangle$, at which point the $n = 2$ case will finish the proof.

So suppose $F \in \langle f_1, \ldots, f_{n+1} \rangle$. Then there exist entire functions $\langle g_1, \ldots, g_{n+1} \rangle$ so that $F = \sum_{j=1}^{n} f_j g_j + f_{n+1} g_{n+1}$. On the right-hand side, the sum from 1 to n belongs to $\langle f_1, f_2, \ldots, f_n \rangle$, so has the form φe for some $e \in \mathscr{E}$. Thus

$$F \in \langle \varphi, f_{n+1} \rangle, \text{ hence } \langle f_1, \ldots, f_{n+1} \rangle \subset \langle \varphi, f_{n+1} \rangle.$$

Conversely, if $F \in \langle \varphi, f_{n+1} \rangle$, then there exist entire functions γ and g_{n+1} such that $F = \varphi\gamma + f_{n+1} g_{n+1}$. Now φ, by its choice of zero-list, belongs to $\langle f_1, f_2, \ldots, f_n \rangle$ there exist entire functions $\gamma_1, \gamma_2, \ldots, \gamma_n$ with $\varphi = \sum_{j=1}^{n} f_j \gamma_j$. Consequently, if $g_j = \gamma_j \gamma$ for $1 \leq j \leq n$, then

$$F = \varphi\gamma + f_{n+1} g_{n+1} = \sum_{j=1}^{n+1} f_j g_j\,,$$

hence

$$F \in \langle f_1, \ldots, f_{n+1} \rangle \text{ hence } \langle \varphi, f_{n+1} \rangle \subset \langle f_1, \ldots, f_{n+1} \rangle,$$

which establishes the desired equality. □

7.5 *Closed ideals in $\mathscr{E}$*

It's well-known that analyticity is preserved by uniform convergence on compact sets. More precisely—in our setting—if (f_n) is a sequence of entire functions, and $f_n \to f$ uniformly on each compact subset of the plane, then f is also entire.

We'll say that an ideal $\mathscr{I}$ in $\mathscr{E}$ is *closed* if: whenever a sequence of functions in $\mathscr{I}$ converges uniformly on every compact subset of the plane, the limit function (which we just observed belongs to $\mathscr{E}$) also belongs to $\mathscr{I}$.

Proposition 7.5. *Every finitely generated ideal in $\mathscr{E}$ is closed.*

Proof. Suppose $\mathscr{I}$ is finitely generated ideal in $\mathscr{E}$. By Theorem 7.4 that $\mathscr{I}$ is principal, i.e., singly generated. Let g be a generator of $\mathscr{I}$, so $\mathscr{I} = g\mathscr{E}$. Suppose (f_n) is a sequence in $\mathscr{I}$ that is uniformly convergent on compact subsets of the plane to a function f, necessarily in $\mathscr{E}$. We wish to show that $f \in \mathscr{I}$.

For each index n we have $f_n = ge_n$ for some $e_n \in \mathscr{E}$. We'll be done if we can show that the sequence e_n converges uniformly on compact subsets of the plane to some function e. Then we'll have $e \in \mathscr{E}$, and $f = ge$, as desired. To this end, fix K a compact subset of the plane, and γ a circle in the plane centered at the origin that does not intersect any zeros of g. Let $m = \min_{z\in\gamma} |g(z)|$, so $m > 0$ (by the continuity of g). Consequently, at each point of γ we have for each pair m, n of indices:

$$|e_n - e_m| = \frac{1}{|g|}\,|f_n - f_m| \le \frac{1}{m}|f_n - f_m|$$

hence, because (f_n) is Cauchy uniformly on γ, so is (e_n). The Maximum Principle now insures that (e_n) is Cauchy uniformly con K, and so converges there (uniformly).

Conclusion. (e_n) converges uniformly on each compact subset of $\mathbb{C}$ to an entire function e, hence $f = ge \in \langle g\rangle = \mathscr{I}$, as desired. □

Theorem 7.6. *Every closed ideal in $\mathscr{E}$ is principal.* [3]

[3] See Schilling [7, 1946.]

Proof. Suppose $\mathscr{I}$ is a closed ideal of $\mathscr{E}$. We break the proof into several steps.

STEP I. $\mathscr{I}$ has no common zero. We'll show that in this case $\mathscr{I} = \mathscr{E}$. To this end, fix $g \in \mathscr{I}$.

(a) If g has no zeros then $1 = g^{-1}g \in \mathscr{I}$, so $\mathscr{I} = \mathscr{E}$.

(b) Suppose g has just finitely many distinct zeros $a_1, a_2, \ldots, a_n$ (we list them now *without* multiplicity). Since $\mathscr{I}$ has no common zero, for each index k there will exist $f_k \in \mathscr{I}$ with $f_k(a_k) \neq 0$. Choose an entire function f having a simple zero at each a_k, and no other zeros. E.g., $f(z) = \prod_{k=1}^{n}(z - a_k)$ will work. Then for each index k,

$$h_k(z) = \frac{f(z)f_k(z)}{z - z_k} \tag{24}$$

defines a function h_k that is the product of $f_k \in \mathscr{I}$ with $f/(z - z_k) \in \mathscr{E}$. Thus $h_k \in \mathscr{I}$, so the same is true of $h = \sum_{k=1}^{n} h_k$. Moreover: h_k take the value zero at each zero of g, *except for a_k*, where its value is*not* zero. Thus h takes non-zero values at each zero of g, so we have produced two functions in $\mathscr{I}$ (namely g and h) with no common zero. From Theorem 7.2, $\mathscr{E} = \langle g, h\rangle$, and since $\langle g, h\rangle \subset \mathscr{I}$, we must have $\mathscr{E} = \mathscr{I}$, as desired.

(c) Suppose, finally, that our chosen function g in $\mathscr{I}$ has infinitely many zeros $(a_1, a_2, \ldots)$. By Corollary 6.2 (of Mittag-Leffler's Theorem) we know that there is an entire function f with a simple zero at each zero a_k of g, and no other zero. As in the previous case, our assumption that $\mathscr{I}$ has no common zero insures that for each index k there exists $f_k \in \mathscr{I}$ with $f_k(z_k) \neq 0$, hence—as in part (b) above—the function h_k, as defined by (24), belongs to $\mathscr{I}$ and takes the value 0 at each zero of g, except for a_k, where it's not zero.

Since we have infinitely many h_k's, we can't just sum them up. But upon multiplying each of them by an appropriate positive constant, we may assume that $|h_k(z)|$ is less than 2^{-k} on the disc $\{|z| \leq k\}$. With this scaling (which preserves the zero-behavior of the original h_k), the series $\sum_k h_k$ converges uniformly on compact subsets of the plane to a function h which, *since $\mathscr{I}$ is assumed to be closed*, belongs to $\mathscr{I}$, and which has no zero in common with g. Thus, just as in part (b), we've produced two functions g and h in $\mathscr{I}$ which have no common zero, so just as before $\mathscr{I} = \mathscr{E}$. This completes the proof of STEP I.

STEP II. This one is familiar. Suppose the list $\mathscr{Z}$ of common zeros of $\mathscr{I}$ is not empty. Let g denote an entire function which, Corollary 6.2 again, $Z(g) = Z$, i.e., it has as its zero-list the common zero -list of $\mathscr{I}$. Then $\frac{1}{g}\mathscr{I} = \{f/g \colon f \in \mathscr{I}\}$ is a closed ideal of $\mathscr{E}$ with no common zero, so it is all of $\mathscr{E}$. Thus $\mathscr{I} = g\mathscr{E} = \langle g \rangle$, as desired. $\square$

The *closure* of a set $\mathscr{S} \subset \mathscr{E}$ is the set of all functions in $\mathscr{E}$ that are limits—uniformly on compact subsets of the plane—of sequences drawn from $\mathscr{S}$. We say $\mathscr{S}$ is *dense* in its closure.

It's easy to see that the closure of an ideal in $\mathscr{E}$ is also an ideal. Thus:

Corollary 7.7. *If an ideal in $\mathscr{E}$ has no common zero, then it is dense in $\mathscr{E}$.*

For instance: Example(c) on page 13, an ideal in $\mathscr{E}$ that has no common zero, is not all of $\mathscr{E}$, so it is dense, but not closed.[4]

[4] *Exercise*. Give direct proofs of these statements.

References

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