THE SCHROEDER-BERNSTEIN THEOREM VIA FIXED POINTS

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ABSTRACT. These notes discuss the Knaster-Tarski fixed point theorem, and its application to proving the Schroeder-Bernstein theorem.

1. Introduction

According to the Schroeder-Bernstein theorem, if two sets have the property that each has a map taking it injectively into the other, then the sets are bijectively equivalent. In other words: if the cardinality of each set is less than or equal to that of the other, then the sets have the same cardinality. The result follows quickly from something more general:

The Banach Decomposition Theorem. *Suppose X and Y are sets, and we are given maps $f\colon X \to Y$ and $g\colon Y \to X$. Then there is a subset A of X such that g that maps $Y \backslash f(A)$ onto $X \backslash A$.*

Proof of the Schroeder-Bernstein Theorem. Suppose, in the statement of the Banach Decomposition Theorem that the maps f and g are injective (i.e. one-to-one). Then the map $h\colon X \to Y$ defined by setting $h = f$ on A and $h = g^{-1}$ on $X \backslash A$ is the desired bijection taking X onto Y. □

It remains to prove the Banach Decomposition Theorem. This will follow quickly from a surprisingly simple, but very important, fixed point theorem.

2. The Knaster-Tarski fixed point theorem

A *partial order* on a set X is a binary relation "$\leq$" such that for all $x, y, z \in X$:

$x \leq x$ (reflexivity),
$x \leq y$ and $y \leq x \implies x = y$ (antisymmetry), and
$x \leq y$ and $y \leq z \implies x \leq z$ (transitivity).

Date: November 3, 2012.

Notes based on a lecture delivered by John Erdman during the 2012 Fall term in the Analysis Seminar at Portland State University. Any errors herein are solely my responsibility.

If "$\leq$" is a partial order on X then the pair $(X, \leq)$ is called a *poset* (short for "partially ordered set"). Suppose $(X, \leq)$ is a poset and $S \subset X$. An element $b \in X$ for which each $s \in S$ is $\leq b$ is called an *upper bound* for S. If such an element is $\leq$ any other upper bound for S it's called a *least upper bound* for S. By antisymmetry, there can be at most one such element, which we denote (when it exists) by $\sup S$ (for "supremum of S"). Similarly one can define "lower bound" and "greatest lower bound" (or "inf") for subsets of X.

A poset $(X, \leq)$ is said to be a *lattice* if every pair of elements of X has a least upper bound and a greatest lower bound. If this is true more generally for every *subset* of X, then the lattice is said to be *complete.* In this case X has a largest element ($\sup X$) and a smallest one ($\inf X$).

If $(X, \leq)$ is a poset, then a map $f : X \to X$ is said to be *order preserving* (on the poset) if, for every pair x, y of elements of X with $x \leq y$, we have $f(x) \leq f(y)$.

The Knaster-Tarski Fixed Point Theorem. *Every order preserving mapping on a complete lattice has a fixed point.*

Proof. Suppose $(X, \leq)$ is our complete lattice and $f : X \to X$ is the order preserving map. Let

$$E \;=\; \{x \in X : x \leq f(x)\}$$

(non-empty because it contains $\inf X$), and set $s = \sup E$. Then for each $x \in E$ we have $x \leq f(x)$ by the definition of E , and $f(x) \leq f(s)$ by the definition of s and the order preserving nature of f. Thus, by transitivity, $f(s)$ is an upper bound for E, hence $s \leq f(s)$. Again, by order-preservingness, $f(s) \leq f(f(s))$, so $f(s) \in E$, hence $f(s) \leq s$. Thus, by antisymmetry, $f(s) = s$. $\square$

Remarks on the proof. (a) In the above proof, every fixed point of f belongs to E, so $s = \sup E$ is the *largest* fixed point of f. If we replace the set E of the proof by $E_0 := \{x \in X : f(x) \leq x\}$ then a similar argument shows that $\inf E_0$ is a fixed point—in fact the *smallest* fixed point—of f. With a little bit more work it's possible to show that: *if F is the set of fixed points of f, then $(F, \leq)$ is a complete lattice* (see [1, Theorem 1.2, Page 1], for example).

(b) Our proof of the Knaster-Tarski theorem required only that $(X, \leq)$ be a partially ordered set for which every subset had a supremum, and—to insure that the set E in the proof was nonempty—that X *have a least element.* However nothing is gained by stating the theorem under this apparently weaker assumption since: *Every such partially ordered set is a complete lattice.*

Proof (see, for example, [1, Lemma 1.3, pp. 1–2]). Suppose A is a non-empty subset of X. Our goal is to show that A has a greatest lower bound. To aid the discussion, let's denote the least element of X by $\mathbb{O}$, and for $a \in X$ denote by $[\mathbb{O}, a]$ the set of all elements x in X that are $\leq a$. I claim that $b := \sup \cap_{a\in A}[\mathbb{O}, a]$ is the greatest lower bound of A. Clearly $b \leq \sup[\mathbb{O}, a] = a$ for every $a \in A$, so b is a lower bound for A. On the other hand, if β is *any* lower bound for A then $\beta \in [\mathbb{O}, a]$ for every $a \in A$, i.e. $\beta \in \cap_{a\in A}[\mathbb{O}, a]$, and so $\beta \leq b$. Thus $b = \inf A$, as desired. □

3. Proof of the Banach Decomposition

For a set X let $\mathscr{P}(X)$ denote the power set of X, i.e. the collection of all subsets of X. When endowed with the usual set inclusion, $\mathscr{P}(X)$ becomes a complete lattice—the supremum of a collection of its elements (i.e. a collection of subsets of X) being just the union of those elements, and the corresponding infimum being the intersection. Given sets X and Y, with maps $f\colon X \to Y$ and $g\colon Y \to X$, define $\Phi\colon \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$\Phi(E) = X\backslash g(Y\backslash f(E)) \qquad (E \in \mathscr{P}(X)),$$

i.e., define Φ by following the diagram below around clockwise from the upper left-hand corner—where the maps C_X and C_Y denote complementation in X and Y respectively, and where f and g are now regarded as mappings of sets:

$$\begin{tikzcd}
\mathscr{P}(X) \arrow[r, "f"] & \mathscr{P}(Y) \arrow[d, "C_Y"] \\
\mathscr{P}(X) \arrow[u, "C_X"] & \mathscr{P}(Y) \arrow[l, "g"]
\end{tikzcd}$$

Thus Φ, being the composition of two maps that are order preserving (f and g), and two maps that are order *reversing* (the complementation maps), is itself order preserving. Hence, by the Knaster-Tarski Theorem, Φ has a fixed point $A \in \mathscr{P}(X)$. That is, $A = X\backslash g(Y\backslash f(A))$; the desired result now follows upon complementing both sides of this equality. □

4. Closing remarks

Historical notes. In [5, page 286] Tarski points out that in the 1920's he and Knaster discovered their fixed point theorem in the setting of order preserving maps of power sets, with Knaster publishing the result in [4]. Tarski goes on to say that he found the generalization presented here, and lectured on it and its applications, during the late 1930's and and early 1940's before finally publishing his results in [5].

Fixed-point characterization of completeness. Tarski posed the question of whether or not the fixed point property guaranteed by the Knaster-Tarski theorem actually *characterizes* completeness for lattices. His student Davis gave an affirmative answer in [2, Theorem 2, page 313]: *If a lattice has the property that every order preserving mapping has a fixed point, then that lattice is complete.* The preprint [1] contains a nice exposition of the Knaster-Tarski theorem, some of its applications, and Davis's converse.

Fixed points via iteration. If we demand that the order preserving function $f\colon X \to X$ of the Knaster-Tarski theorem also be *sequentially continuous* in the sense that $f(\sup_n x_n) = \sup_n f(x_n)$ for every increasing sequence $x_1 \le x_2 \le \dots$ of elements of X, then a bit more can be said. In what follows: for a mapping $f\colon X \to X$ and a natural number n, we denote by $f^{(n)}$ the composition of f with itself n times.

Theorem. *Suppose $(X, \le)$ is a complete lattice with smallest element $\mathbb{O}$, and suppose $f\colon X \to X$ is order preserving and sequentially continuous. Then the smallest fixed point of f is $\sup_{n\in\mathbb{N}} f^{(n)}(\mathbb{O})$.*

Proof. We have

$$\mathbb{O} \;\le\; f(\mathbb{O}) \;\le\; f^{(2)}(\mathbb{O}) \;\le\; \dots ,$$

where the first inequality follows from the minimality of the element $\mathbb{O}$, and the succeeding ones from this and the order preserving nature of the mapping f. Thus if $p := \sup_{n\in\mathbb{N}} f^{(n)}(\mathbb{O})$ then the continuity of f yields $f(p) = \sup_{n\in\mathbb{N}} f^{(n+1)}(\mathbb{O})$. But the latter supremum is just p, since by the increasingness of the sequence $(f^{(n)}(\mu))$ the supremum of the set $\{f^{(n)}(\mathbb{O})\colon n \ge 2\}$ is the same as that of $\{f^{(n)}(\mathbb{O})\colon n \ge 1\}$. So p is a fixed point of f. As for its minimality, suppose q is any fixed point of f. Then by the minimality of $\mathbb{O}$ in X and the order preserving nature of f we know that $f^{(n)}(\mathbb{O}) \le q$ for each $n \in \mathbb{N}$. Thus $p := \sup f^{(n)}(\mathbb{O}) \le q$. $\square$

Fundamental theorem of computer science? A closely related version of the Knaster-Tarski theorem (see [3, Theorem 2.4, page 3]) seems to play an important role in theoretical aspects of computer science. From [3, Theorem 2.4 ff]:

> *This theorem has so many applications to computing that it must be a contender for the title of fundamental theorem of computer science.*

References

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[4] B. Knaster, *Un théorème sur les fonctions d'ensembles,* Ann. Soc. Polon. Math. 6 (1928) 133–134.

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