Sperner's Lemma and Brouwer's Fixed-Point Theorem

Joel H. Shapiro

April 12, 2015

Abstract. These notes present a proof of the Brouwer Fixed-Point Theorem using a remarkable combinatorial lemma due to Emanuel Sperner. The method works in $\mathbb{R}^N$ for all N, but for simplicity we'll restrict the discussion to $N = 2$.

Overview. In dimension two the Brouwer Fixed-Point Theorem states that every continuous mapping taking a closed disc into itself has a fixed point. Here we'll give a proof of this special case of Brouwer's result, but for triangles, rather than discs; closed triangles are homeomorphic to closed discs (Exercise 2.1 below) so our result will be equivalent to Brouwer's. We'll base our proof on an apparently unrelated combinatorial lemma due to Emanuel Sperner, which—in dimension two—concerns a certain method of labeling the vertices of "regular" decompositions of triangles into subtriangles. We'll give two proofs of this special case of Sperner's Lemma, one of which has come to serve as a basis for algorithms designed to approximate Brouwer fixed points.

1 *Sperner's Lemma*

Throughout this discussion, "triangle" means "closed triangle," the convex hull of three points in Euclidean space that don't all lie on the same straight line. A "regular decomposition" of a triangle is a collection of subtriangles whose union is the original triangle and for which the intersection of any two distinct subtriangles is either a vertex or a complete common edge. Figure 1 below illustrates a regular and an irregular decomposition of a triangle into subtriangles.

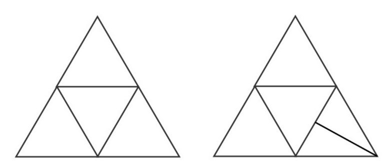

Figure 1: *Regular (left) and irregular (right) decomposition of a triangle.*

A "Sperner Labeling" of the subvertices (the vertices of the subtriangles) in a regular decomposition is an assignment of labels "1", "2", or "3" to each subvertex in such a way that:

(a) No two vertices of the original triangle get the same label (i.e., all three labels get used for the original vertices),

(b) Each subvertex lying on an edge of the original triangle gets labels drawn only from the labels of that edge, e.g. subvertices on the original edge labeled "1" and "2" (henceforth: a "$\{1, 2\}$ edge") get only the labels "1" or "2", but with no further restriction. Subvertices lying in the interior of the original triangle can be labeled without any restrictions.

We'll call a subtriangle whose vertices have labels "1", "2", and "3" a *completely labeled subtriangle*. Figure 2 shows a triangle regularly decomposed into Sperner-labeled subtriangles, five of which (the shaded ones) are completely labeled.

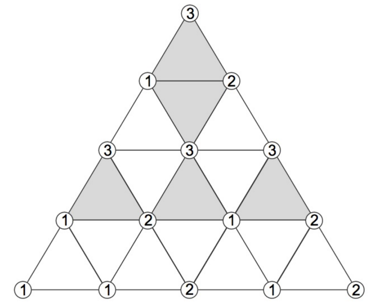

Figure 2: *A Sperner-labeled regular decomposition into subtriangles*

Theorem 1.1 (Sperner's Lemma for dimension 2)**.** *In a Sperner-labeled regular decomposition of a triangle there is at least one completely labeled subtriangle; in fact, there is an odd number of them.*

The one dimensional case. Here, instead of triangles split "regularly" into subtriangles, we just have a closed finite line segment split into closed sub-segments, any two of which intersect in at most a common endpoint. One end of the original segment is labeled "1" and the other is labeled "2". The remaining subsegment endpoints get these labels in any way whatever.

Sperner's Lemma for this situation asserts that: *There is an odd number of subsegments (in particular, at least one!) whose endpoints get different labels.*

To prove this let's imagine moving from the 1-labelled endpoint of our initial interval toward the 2-labelled one. If there are no subintervals, we're done. Otherwise there has to be a first subinterval endpoint whose label switches from "1" to "2", thus yielding a completely labelled subinterval with final endpoint "2". At the next switch, if there is one, the initial endpoint is "2" and the final endpoint is "1", thus yielding another completely labelled subinterval which must, somewhere further on the line have an oppositely labelled companion (else we'd never be able to end up with the final subinterval labelled "2"). Thus there must be an odd number of completely labelled subintervals. □

The two dimensional case. We start with a triangle Δ regularly decomposed into a finite collection of subtriangles $\{\Delta_j\}$. Let $\nu(\Delta_j)$ denote the number of "$\{1,2\}$ edges" belonging to the boundary of Δ_j, and set $S = \sum_j \nu(\Delta_j)$.

CLAIM. *S is odd.*

Proof of Claim. If a $\{1,2\}$ edge of Δ_j does not belong to the boundary of Δ then it belongs to exactly one other subtriangle. If a $\{1,2\}$ edge of Δ_j lies on the boundary of Δ, then that edge belongs to no other subtriangle. Thus S is twice the number of "non-boundary" $\{1,2\}$ edges plus the number of "boundary" $\{1,2\}$ edges. But by the one dimensional Sperner Lemma, the number of boundary $\{1,2\}$ edges is odd. Thus S is odd.

Completing the proof of Sperner's Lemma. Note that the $\{1,2\}$ edges of subtriangles occur just once on the boundary of each completely labelled subtriangle, twice on the boundaries of triangles whose vertices are labeled with just 1 and 2, and not at all in every other case. Thus the odd number S is the number of completely labeled triangles plus twice the number of subtriangles with $\{1,2\}$ edges, from which we conclude that our Sperner-labeled regular decomposition of Δ has an odd number of completely labeled subtriangles. $\square$

2 *Proof of Brouwer's Theorem for a triangle*

We may assume, without loss of generality (see the exercise below), that our triangle Δ is the the standard 3-simplex; the set of vectors $x \in \mathbb{R}^3$ whose coordinates are non-negative and sum to 1 (i.e. the convex hull of the standard unit vectors in $\mathbb{R}^3$).

Exercise 2.1. Show that every closed triangle is homeomorphic to a closed disc.

> *Suggestion:* First argue that without loss of generality we can suppose that our triangle T lies in $\mathbb{R}^2$, contains the origin in its interior, and is contained in the closed disc D of radius 1 centered at the origin. Then each point $z \in T\backslash\{0\}$ is uniquely represented as $z = r\zeta$ for $\zeta \in \partial D$ and $r > 0$. Let $w = \rho\zeta$ be the point at which the line through the origin and z intersects ∂T. Show that the map that takes the origin to itself and $z \neq 0$ to $(r/\rho)\zeta$ is a homeomorphism of T onto D.

Fix a continuous self-map f of Δ; for each $x \in \Delta$ write

$$f(x) = (f_1(x), f_2(x), f_3(x)).$$

Thus for each index $j = 1,2,3$ we have a continuous "coordinate function" $f_j \colon \Delta \to [0,1]$ with

$$f_1(x) + f_2(x) + f_3(x) = 1$$

for each $x \in \Delta$.

A Sperner labelling induced by f. Consider a regular decomposition of Δ into subtriangles and suppose f fixes no subvertex. Then f determines a Sperner labeling of subvertices. Here's how! Fix a subtriangle vertex p. Since $f(p) \neq p$, at least one coordinate of $f(p)$ is *dominated*, i.e. strictly less than the corresponding coordinate of p.

> Indeed, since we are assuming that $f(p) \neq p$, some coordinate of $f(p)$ is not equal to the corresponding one of p. If it's strictly less than the corresponding coordinate of p, we're done. Otherwise it's strictly greater than that coordinate, so in order for all coordinates of $f(p)$ to sum to 1, some other coordinate of $f(p)$ must be strictly less than the corresponding one of p (otherwise $\sum_j f_j(p) > \sum_j p_j = 1$, contradicting the fact that $f(p) \in \Delta$).

Choose a dominated coordinate; label the subvertex p with its index.

In this way the three original vertices $e_1 = (1,0,0), e_2 = (0,1,0)$, and $e_3 = (0,0,1)$, get the labels "1", "2", and "3" respectively; for example $f(e_1) \neq e_1$, so the first coordinate of $f(e_1)$ must be strictly less than 1 (else $f(e_1)$ would have to equal e_1), and similarly for the other two vertices of Δ. Any vertex on the $\{1,2\}$ edge of $\partial\Delta$ (the line segment joining e_1 to e_2) has third coordinate zero, so this coordinate cannot strictly decrease when that vertex is acted upon by f. Thus (since that vertex is not fixed by f) at least one of the other coordinates must strictly decrease, so each vertex on the$\{1,2\}$ edge gets only the labels "1" or "2", as required by Sperner labeling. Similarly for the other edges of $\partial\Delta$; the vertices on the (2,3)-edge get only labels "2" and "3", and the vertices on the (1,3)-edge get only labels "1" and "3". No further checking is required for the labels induced by f on the interior vertices; Sperner labeling places no special restrictions here. In this way f determines a Sperner labeling of the vertices of subtriangles in any regular subdivision of Δ. (Note that the continuity assumed for f has not yet been used.)

Approximate fixed points for f. Let $\varepsilon > 0$ be given. We're going to show that our continuous self-map f of Δ has an *ε-approximate fixed point*, i.e. a point $p \in \Delta$ such that $\|f(p) - p\|_1 \leq \varepsilon$. Being continuous on the compact set Δ, the mapping f is *uniformly* continuous there, so there exists $\delta > 0$ such that $x, y \in \Delta$ with $\|x - y\|_1 < \delta$ implies $\|f(x) - f(y)\|_1 < \varepsilon/8$. Upon decreasing δ if necessary we may assume that $\delta < \varepsilon/8$. Now suppose Δ is regularly decomposed into subtriangles of $\|\cdot\|_1$-diameter $< \delta$, with no subvertex a fixed point of f. Thus f creates a Sperner labelling of the subvertices of this decomposition. Let Δ_ε be a completely labelled subtriangle as promised by Sperner's Lemma.

Here $\|v\|_1$ is the "1-norm" of $v \in \mathbb{R}^3$: the sum of the absolute values of the coordinates of v.

If some subvertex is a fixed point of f, we're done!

CLAIM. *Δ_ε contains an ε-approximate fixed point.*

Proof of Claim. Let p, q, and r be the vertices of Δ_ε, carrying the labels "1", "2", and "3" respectively, so that $f_1(p) < p_1$, $f_2(q) < q_2$, and $f_3(r) < r_3$. Thus:

$$\begin{aligned}
\|p - f(p)\|_1 &= \underbrace{p_1 - f_1(p)}_{>0} + |p_2 - f_2(p)| + |p_3 - f_3(p)| \\
&= p_1 - f_1(p) + |q_2 - f_2(q) + p_2 - q_2 + f_2(q) - f_2(p)| \\
&\qquad\qquad + |r_3 - f_3(r) + p_3 - r_3 + f_3(r) - f_3(p)| \\
&\leq \underbrace{p_1 - f_1(p)}_{>0} + \underbrace{q_2 - f_2(q_2)}_{>0} + \underbrace{r_3 - f_3(r_3)}_{>0} \\
&\qquad\qquad + |p_2 - q_2| + |f_2(q) - f_2(p)| \\
&\qquad\qquad + |p_3 - r_3| + |f_3(r) - f_3(p)|,
\end{aligned}$$

so $\|p - f(p)\|_1 \leq A + B$, where

$$A := [p_1 - f_1(p)] + [q_2 - f(q_2)] + [r_3 - f(r_3)],$$

which is > 0 since this is true of each bracketed term, and

$$B := |p_2 - q_2| + |f_2(q) - f_2(p)| + |p_3 - r_3| + |f_3(r) - f_3(p)|. \tag{1}$$

Now each summand on the right-hand side of (1) is $< \varepsilon/8$, hence $B < \varepsilon/2$. As for A, the same "adding-zero trick" we used above yields

$$\begin{aligned} A = \underbrace{p_1 + p_2 + p_3}_{=1} - \underbrace{f_1(p) + f_2(p) + f_3(p)}_{=1} \\ + [q_2 - p_2] + [f_2(p) - f_2(q)] \\ + [r_3 - p_3] + [f_3(p) - f_3(r)]. \end{aligned}$$

On the right-hand side of this equation, the top line equals zero and each bracketed term has absolute value $< \varepsilon/8$, so by the triangle inequality, $A < \varepsilon/2$. These estimates on A and B yield $\|p - f(p)\| < \varepsilon$, i.e. the vertex p of Δ_ε is an ε-approximate fixed point of f. □

The same argument shows that the other two vertices of Δ_ε are also ε-approximate fixed points of f; the triangle inequality shows that *every* point of Δ_ε is a $\frac{5}{4}\varepsilon$-approximate fixed point.

A fixed point for f. So far we know that each continuous self-map of a (closed) triangle has an ε-approximate fixed point for every $\varepsilon > 0$. That this implies our map has an actual fixed point is a special case of the following lemma.

Lemma 2.2 (The Approximate Fixed-Point Lemma). *Suppose (X, d) is a compact metric space and $f\colon X \to X$ is a continuous map. Suppose that for every $\varepsilon > 0$ there exists a point $x_\varepsilon \in X$ with $d(f(x_\varepsilon), x) \leq \varepsilon$. Then f has a fixed point.*

Proof. We're given that for each positive integer n there exists $x_n \in X$ such that $d(f(x_n), x_n) < 1/n$. Since X is compact there is a subsequence $n_k \nearrow \infty$ and a point $y \in X$ such that $y_k := x_{n_k} \to y$. By continuity $f(y_k) \to f(y)$, hence by the continuity of the metric:

$$d(y, f(y)) = \lim_k d(y_k, f(y_k)) \leq \lim_k (1/n_k) = 0,$$

so $y = f(y)$. □

Exercise. Here's another way to produce fixed points from completely labelled subtriangles. Make a regular decomposition of Δ into subtriangles of diameter $< 1/n$. For this decomposition of Δ, use f to Sperner-label the subvertices, and let Δ_n be a resulting completely labelled subtriangle. Denote the vertices of Δ_n by $p^{(n)}$, $q^{(n)}$, and $r^{(n)}$, using the previous numbering scheme so that $f_1(p^{(n)}) \leq p_1^{(n)}$, etc.

Show that it's possible to choose a subsequence of integers $n_k \nearrow \infty$ such that the corresponding subsequences of p's, q's, and r's all converge. Show that these three subsequences all converge to the same point of Δ, and that this point is a fixed point of f.

3 *Finding fixed points by "walking through rooms"*

Finding fixed points "computationally" amounts to finding an algorithm that produces sufficiently accurate approximate fixed points. Thanks to the work just done in §2, what's needed is an algorithm for finding a completely labeled subtriangle. Here's an alternate proof of Sperner's Lemma that speaks to this issue.

Suppose we have a closed triangle Δ regularly decomposed into subtriangles with the subvertices given a Sperner labelling. Imagine that Δ is a house, that its subtriangles are rooms, that each $\{1,2\}$-labeled segment of a subtriangle boundary is a door, and there are no other doors. For example a $\{1,2,2\}$-labeled subtriangle has two doors, some rooms have no doors (e.g., those with no sub-vertex labeled "2"), and the completely labeled subtriangles are those rooms with exactly one door.

Now imagine that you are outside the house. *There is a door to the inside*; the Sperner labeling of the sub-vertices induces on the original $\{1,2\}$ edge a one dimensional Sperner labeling, which must produce a $\{1,2\}$-labeled subinterval. Go through this door. Once inside, either the room you're in has no further door, in which case you're in a completely labeled subtriangle, or there is another door to walk through. Keep walking, subject to the rule that you can't pass through a door more than once (i.e. the doors are "trap-doors"). There are two possibilities. Either your walk terminates in a completely labeled room, in which case you're done, or it doesn't in which case you find yourself back outside the house. In that case, you've used up two doors on the $\{1,2\}$ edge of the original triangle: one to go into the house, and the other to come back out. But according to the one dimensional Sperner Lemma, there are an odd number of such doors, so there's one you haven't used, through which you can re-enter the house. Continue. In a finite number of steps you must encounter a room with just one door—a completely labeled one.

Figure 3 illustrates this process. Starting at point A one travels through through three rooms, arriving outside at point B, in which case the process starts again, this time terminating in C, a completely labelled triangle (the only one for this particular labeling).

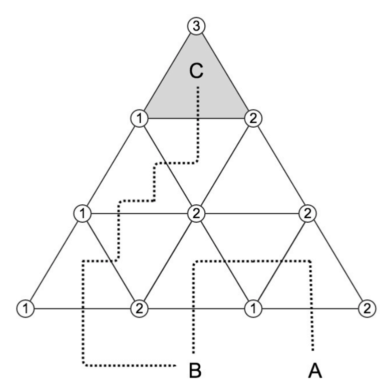

Figure 3: *Finding a completely labeled subtriangle by walking through rooms.*

Notes

The Brouwer Fixed-Point Theorem.[1] Brouwer proved this result using topological methods of his own devising. It is one of the most famous and widely applied theorems in mathematics; see Park[2] for an exhaustive survey of the legacy of this result, and Casti[3] for a popular exposition.

[1] Brouwer, L.E.J.: Math. Ann. **71**, 97–115 (1912)

[2] Park, S.: Vietnam J. Math. **27**, 187–222 (1999)

[3] Casti, J.L.: Five Golden Rules... John Wiley & Sons (1965)

Sperner's Lemma, higher dimensions. This result for all finite dimensions appears in Sperner's 1928 doctoral dissertation[4]. In dimensions > 2 the analogue of a triangle is an "N-simplex" in $\mathbb{R}^N$; the convex hull of $N+1$ points of $\mathbb{R}^N$ in "general position," i.e. no point belongs to the convex hull of the others, and the analogue of our regular decomposition of a triangle is a "triangulation" of an N-simplex into "elementary sub-simplices," each of which is itself an N-simplex.

[4] Sperner, E.: Abh. a.d. Math. Sem. d. Univ. Hamburg **6**, 265–272 (1928)

Nice descriptions of this generalization occur in Franklin s book[5], and in E. F. Su's expository article[6], which also provides a proof of the general Brouwer theorem based on "walking through rooms." Su's article also contains interesting applications of Sperner's Lemma to problems of "fair division."

[5] Franklin, J.: Methods of Mathematical Economics. Springer-Verlag (1980)

[6] Su, F.E.: Amer. Math. Monthly **106**, 930–942 (1999)

Walking through rooms. The argument seems to have its origin in a 1965 paper of Lemke[7]. This technique has been greatly refined to produce useful algorithms for finding approximate fixed points, especially by Scarf, whose survey[8] introduces the reader to the way in which economists view Brouwer's theorem, and provides a nice introduction to the algorithmic search for fixed points.

[7] Lemke, C.E.: Management Science (Ser. A, Sciences) **11**, 681–689 (1965)

[8] Scarf, H.E.: American Scientist **71**, 289–296 (1983)

Fariborz Maseeh Department of Mathematics & Statistics
Portland State University
PO Box 751, Portland, OR 97207-0751
`joels314(at)gmail.com`
`www.joelshapiro.org`