The Prime-Number Theorem

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The Prime-Number Theorem, perhaps the most spectacular result of 19th century mathematics, tells us that the size of the n-th prime number is approximately $n \log n$. These notes present a modern proof that offers an instructive journey through elementary aspects of both real and complex analysis.

Notation

- *Sets of numbers*. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{P}$ denote, respectively; the complex plane, the real line, the positive integers (a.k.a. "natural numbers"), and the set of primes (2, 3, 5, 7, ...).

- *Integer / fractional part*. For $x \in \mathbb{R}$: we'll denote by $[x]$ the *integer part*, and by $\{x\}$ the *fractional part*. Thus $x = [x] + \{x\}$.

- *Primes*. p will always denote a "generic" prime. We'll use $\sum_p$ to abbreviate $\sum_{p \in \mathbb{P}}$ We'll denote the n-th prime by p_n (e.g., $p_1 = 2$, $p_6 = 13$).

  For $x \geq 2$ we'll use $\pi(x)$ to denote the number of primes $\leq x$. Thus, e.g., $\pi(3) = 2, \pi(14) = 6$, and $\pi(p_n) = n$.

- *Asymptotics*. For functions f and g on the non-negative real axis, we'll use the notation:
  * $f(x) \sim g(x)$ to mean that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$,
  * $f(x) = o(g(x))$ to mean that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, and
  * $f(x) = O(g(x))$ to mean that there exists $M > 0$ such that $|f(x)| \leq M|g(x)|$ for all sufficiently large x.
  * $f(x) \lesssim g(x)$. Same as $f(x) = O(g(x))$.
  * $f(x) \gtrsim g(x)$: There exists a constant $\delta > 0$ such that $|f(x)| \geq \delta|g(x)|$ for all sufficiently large x. (Same as: $g(x) = O(f(x))$).

- *Complex matters*. Following Riemann, we'll write s for the typical complex number, and $s = \sigma + i\tau$ for its decomposition into real and imaginary parts.

  We'll use the notation "$\{$Property of $s\}$" to describe the set of all $s \in \mathbb{C}$ with that Property. For example, $\{\operatorname{Re} s > 1\}$ denotes the half-plane of complex numbers whose real parts are > 1.

Complex matters won't be discussed until Section 4.

1 Introduction

The "Prime-Number Theorem" (abbreviation: "PNT") asserts—in its classic form— that:

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{i.e., that} \quad \lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1. \tag{1}$$

Carl Friedrich Gauss, 1777-1855

The famous German mathematician Carl Friedrich Gauss conjectured this result in the late 18th century, based on extensive tables of primes that he'd computed in his spare time. However it was not until a century later that Jacques Hadamard in France [7, 1896] and Charles de la Valée Poussin in Belgium [12, 1896] independently proved the result.

In these notes we'll study a modern proof of the PNT, modeled closely on Don Zagier's exposition [15, 1997]. However before embarking on our journey, let's devote some effort to understanding what the PNT wants to tell us.

Size of the n-th prime. As a first step, let's prove that statement (1) is equivalent to the one given above in the Abstract.

Metatheorem 1.1. *The following two statements are equivalent (meaning: the truth of one implies that of the other):*

Here "Metatheorem" means "Theorem about theorems."

(a) The PNT: $\pi(x) \sim \dfrac{x}{\log x}$.

(b) $p_n \sim n \log n$

Proof. (a) → (b): Suppose we've established the PNT in the form (a). The idea now is to "solve" the "asymptotic equation" $y \sim \frac{x}{\log x}$, which we rewrite as

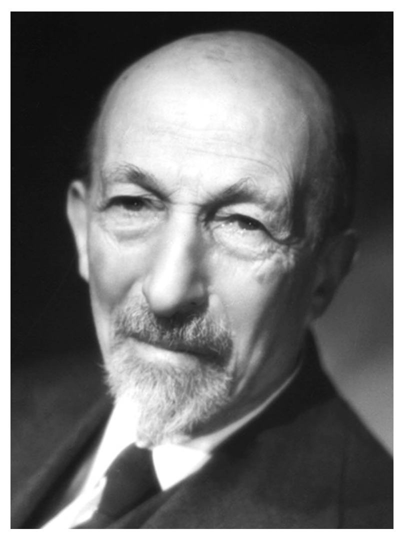
$$y = \left(\frac{x}{\log x}\right)(1 + o(1)) \tag{2}$$

Upon taking logarithms on both sides of (2) we obtain

$$\log y = \log x - \log\log x + \log(1 + o(1)) \sim \log x\,,$$

so from (2) again:

$$x \sim y \log x \sim y \log y\,. \tag{3}$$



Jacques Hadamard, 1865-1963

Thanks to the our assumption (a) we can set $y = \pi(x)$ and $x = p_n$ in (2), thus obtaining $n = \pi(p_n) \sim \frac{p_n}{\log p_n}$ whereupon from (3):

$$p_n \sim \pi(p_n) \log \pi(p_n) = n \log n\,,$$

as desired.

(b) → (a): Now assume that we've established $p_n \sim n \log n$, which for our purposes is best written: $p_n \sim \pi(p_n) \log \pi(p_n)$. As in the previous case, the asymptotic equation $y \sim x \log x$ has asymptotic solution $x \sim y/\log y$, whereupon setting $y = p_n$ and $x = \pi(p_n)$ we obtain

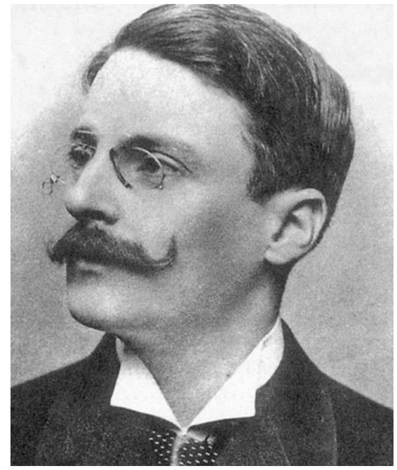
$$n = \pi(p_n) \sim \frac{p_n}{\log p_n}, \tag{4}$$

which establishes (a) for x a prime.

To get the full result, fix $x \geq 2$ and fix the unique n for which $p_n \leq x < p_{n+1}$. As x increases, $\pi(x)$ increases and $(\log x)/x$ decreases (at least for $x \geq e$)), so we have from (4) and $\pi(p_n) = n$:

$$\frac{n}{n+1} \sim \pi(p_n)\frac{\log p_{n+1}}{p_{n+1}} \leq \pi(x)\frac{\log x}{x} < \pi(p_{n+1})\frac{\log p_n}{p_n} \sim \frac{n+1}{n}$$

from which it follows that $\pi(x) \sim \frac{x}{\log x}$, as desired. □



Charles de la Valée Poussin, 1866-1962

As a consequence of the PNT, the sum of the prime reciprocals diverges. More precisely:

Metatheorem 1.2. *The PNT implies that*

$$\sum_{p \leq x} \frac{1}{p} \gtrsim \log\log(x) \qquad (x \to \infty).$$

Proof. For $x \geq 1$:

$$\sum_{p \leq x} \frac{1}{p} = \sum$$

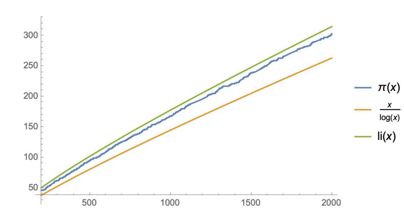
□

The Logarithmic Integral. Statement (1) of the PNT estimates the density of primes in the interval of integers from 1 to n by $1/\log n$. Based on his calculations, Gauss conjectured that the density of primes in any interval about the integer n should be about $1/\log n$, i.e., that the number of primes in an interval $[a, b]$ of the positive real line should be approximately $\int_a^b \frac{1}{\log t}\,dt$. In particular, one might expect the *logarithmic integral*

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\,dt$$

to serve as an asymptotic estimate of $\pi(x)$. This is indeed the case, as we'll now see.

In [13, 1899] de la Vallée Poussin showed that $\mathrm{Li}(x)$ furnishes a *much* better approximation to $\pi(x)$ than does $x/\log x$. The figure below, showing the graphs of $x/\log x$, $\pi(x)$, and $\mathrm{Li}(x)$ (bottom to top) for $200 \leq x \leq 2000$, illustrates this.



Metatheorem 1.3. *The following two statements are equivalent:*

(a) $\pi(x) \sim \dfrac{x}{\log x}$.

(c) $\pi(x) \sim \mathrm{Li}(x)$

Proof. Upon integrating by parts in in the definition of $\mathrm{Li}(x)$ we obtain

$$\mathrm{Li}(x) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{1}{(\log t)^2}\,dt\,. \tag{5}$$

By an application of L'Hospital's rule:

$$\int_2^x \frac{1}{(\log t)^2}\,dt \sim \frac{x}{(\log x)^2} \tag{6}$$

which, along with (5), shows that

$$\left|\mathrm{Li}(x)\frac{\log x}{x} - 1\right| = O\left(\frac{1}{\log x}\right).$$

Thus $\mathrm{Li}(x) \sim x/\log x$, from which follows the equivalence of statements (a) and (c) above. □

THE RIEMANN HYPOTHESIS. The PNT, in the form $\pi(x) \sim \mathrm{Li}(x)$, asserts that the *relative error* that results from approximating $\pi(x)$ by $\mathrm{Li}(x)$ tends to 0 as $x \to \infty$. However, the *actual error* can eventually be large, the present conjecture for its best estimate being $O(\sqrt{x \log x})$. This is one of many equivalent statements of the *Riemann Hypothesis*: open to this day, and widely considered the most important unsolved problem in mathematics.

2 *Euler's Product Formula*

The following result ("Euler's Product Formula) marks the entry of *analysis* into the study of prime numbers.

Theorem 2.1 (Euler 1737). *For each real $s > 1$:*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1} \qquad (s > 1) \tag{7}$$

Leonhard Euler, 1707-1783

Proof. The sum on the right-hand side of Euler's formula is the famous "Riemann zeta-function":

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1) \tag{8}$$

Right now the zeta-function is just a convenient notation, but—thanks to the many consequences of Euler's formula—it has become the most-studied function in all of mathematics.

We have for $s > 1$:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^3} + \frac{1}{4^s} + \cdots,$$

so

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots,$$

and therefore:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots.$$

This is essentially Euler's original argument [5, Thms 7 & 8, pp. 172–176]; see also Dunham [3, Chapter 4], who attributes the rigorous proof to Leopold Kronecker in 1876. Plyman [10, Theorem 3.1.1, p. 14] notes the connection with the "Sieve of Eratosthenes.".

In words: upon multiplying $\zeta(s)$ by $1 - \frac{1}{2^s}$, the result is $1+$ the sum of terms $\frac{1}{n^s}$ where n runs through the odd integers > 1. Similarly: upon multiplying $(1 - \frac{1}{2^s})\zeta(s)$ by $(1 - \frac{1}{3^s})$, we obtain $1+$ the sum of terms $\frac{1}{n^s}$, where now n only runs through all integers > 3 that contain no factor of either 2 or 3. Continuing by induction, we conclude that for each $n \in \mathbb{N}$ (upon writing p_n for the n-th prime),

$$\prod_{k=1}^{N}\left(1 - \frac{1}{p_k^s}\right)\zeta(s) = 1 + \sum\left\{\frac{1}{n^s} : p_1, \dots, p_N \nmid n\right\} \tag{9}$$

Upon using $\Pi_N(s)$ to denote the product on the left-hand side of (9), and $R_N(s)$ to denote the sum on the right, we see that for $s > 1$:

$$0 < \Pi_N(s)\zeta(s) - 1 = R_N(s) < \sum_{n=p_{N+1}}^{\infty} \frac{1}{n^s}.$$

Since $s > 1$, the series on the right $\to 0$ as $N \to \infty$, hence $\Pi_N(s)\zeta(s)$ converges to 1 as $N \to \infty$.

In summary: The infinite product $\prod_p \left(1 - \frac{1}{p^s}\right)$ converges for each $s > 1$ to $1/\zeta(s)$. Since $\zeta(s) > 0$ for each $s > 1$, this proves Euler's formula. □

In fact, the convergence is *uniform* for each $s \geq 1 + \varepsilon$.

The argument just given provides another way of proving:

Corollary 2.2 (Euclid)**.** *There are infinitely many primes.*

Proof. Suppose there were only finitely many primes. Then our argument would have established Euler's Formula (7) after only a finite number of steps, which would then guarantee the (finite) existence of $\lim_{\sigma\to 1+}\zeta(s)$. However for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \geq \int_1^{\infty} \frac{dx}{x^s} = \frac{1}{s-1}. \tag{10}$$

Thus $\zeta(s) \to \infty$ as $s \to 1+$, so there can't be just finitely many primes. □

Maybe comment further that

$$\zeta(s) - 2^{-s} < \int_1^{\infty} x^{-s}dx = 1/(s-1),$$

so $(1-s)\zeta(s) \to 1$ as $s \to 1+$. This prepares the way for Zagier's proof that $\zeta(s)$ extends holo to $\{\mathrm{Re}\, s > 0\}\setminus\{1\}$.

Euler's formula tells us even more.

Corollary 2.3. $\sum_p \frac{1}{p} = \infty$.

In the last section we derived this result from the PNT. Now we see that it would have been enough to use Euler's formula.

Proof. Upon examination of the graphs of $y = x - 1$ and $y = \log x$ we see that $\log x \leq x - 1$ for all $x \geq 1$, whereupon

$$\log\frac{1}{x} \leq \frac{1}{x} - 1 = \frac{1-x}{x} \qquad \forall\, x \in (0,1].$$

Use this inequality with $x = 1 - \frac{1}{p^s}$, where $s \geq 1$, to see that for each $s \geq 1$:

$$\log\left(\frac{1}{1-p^{-s}}\right) \leq \frac{p^{-s}}{1-p^{-s}} \leq 2p^{-s}, \tag{11}$$

with the last inequality due to the fact that

$$1 - p^{-s} \geq 1 - 2^{-s} \geq 1 - 2^{-1} = 1/2 .$$

Upon summing both sides of (11) over all primes and calling on Euler's formula, we see that for $s > 1$:

$$\log \zeta(s) \leq 2 \sum_p p^{-s} \leq 2 \sum_p p^{-1} .$$

The desired result follows upon letting $s \to 1+$ and using (10). □

Some History. The zeta-function has a long and storied. history. The problem of finding $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ in "closed form" is the famous "Basel Problem", was first posed in print in 1644 by the Italian professor Pietro Mengoli . Later in the 17th century, Mengoli's question was popularized by Jacob Bernoulli (at the University of Basel in Switzerland), but it was not until 1735 that the Swiss mathematician Leonhard Euler solved the problem: $\zeta(2) = \pi^2/6$. Euler also found that for $n \in \mathbb{N}$, $\zeta(2n) = \pi^{2n}/b_{2n}$, where the numbers b_{2n} are all positive integers. For example: $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, ... In particular: $\zeta(2n)$ is transcendental for each $n \in \mathbb{N}$.

And again in 1741—this time, rigorously.

However neither Euler, nor anyone since, has been able to determine a closed-form expression for $\zeta(2n+1)$; in fact it's not known to this day if each of these numbers is transcendental. In 1978 Roger Apéry[1] proved $\zeta(3)$ to be *irrational*, and it's recently it has been shown that $\zeta(2n+1)$ is irrational for infinitely many n, but without saying which n's these are.[2]

[1] Apéry [1], see also van der Poorten [14].

[2] See, e.g., [6], where it's observed that the irrationality of $\zeta(5)$ is still open.

3 *Some Prime Number Metatheorems*

In this section we'll present another statement equivalent to the PNT, and see as a corollary that the truth of the PNT follows from the existence of a certain improper integral.

The heavy lifting here was done by the Russian mathematician Pafnuty Chebyshef, who in 1852 showed [4] that $\pi(x)$ and $x/\log x$ tend to infinity at the same rate, and that if their ratio had a limit, then that limit had to be 1.

Pafnuty Chebyshev, 1821-1894

Important to Chebyshev's work, and to all that follows in these notes, was the following "log-weighted" prime counting function.

Definition 3.1 (Chebyshev's ϑ-function)**.**

$$\vartheta(x) = \sum_{p \leq x} \log p \qquad (x \geq 1).$$

Metatheorem 3.2. *The Prime Number Theorem is true if and only if*

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1$$

Here's a first step toward the proof.

Lemma 3.3 (Chebyshev 1852)**.**

$$\vartheta(x) \leq (4\log 2)x \qquad (x \geq 2)$$

Proof. For n a non-negative integer, by the binomial theorem:

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \geq \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

Suppose $n < p \leq 2n$. Then p is a factor of $(2n)!$, but not of $n!$. Thus

$$2^{2n} \geq \prod_{n<p\leq 2n} p\,.$$

whereupon taking logarithms of both sides produces:

$$2n\log 2 \;\geq \sum_{n<p\leq 2n} \log p = \vartheta(2n) - \vartheta(n)\,.$$

It follows that

$$\begin{aligned}\vartheta(2^n) = \vartheta(2^n) - \vartheta(1) &= \sum_{k=1}^{n}[\vartheta(2^k) - \vartheta(2^{k-1})] \\ &\leq \log 2 \sum_{k=1}^{n} 2^k = 2(\log 2)\cdot(2^n - 1)\end{aligned}$$

Thus $\vartheta(2^n) \leq (2\log 2)2^n$ for $n \in \mathbb{N}$, i.e., we've achieved the desired estimate for x a power of 2.

To get the result in general, fix $x \geq 1$ and choose $n \in \mathbb{N} \cup \{0\}$ such that $2^n \leq x < 2^{n+1}$. Then:

$$\vartheta(x) \leq \vartheta(2^{n+1}) \leq (2\log 2)\cdot 2^{n+1} = (4\log 2)\cdot 2^n \leq (4\log 2)\cdot x \qquad \square$$

Proof of Metatheorem 3.2. Write $\pi(x)$ as a Riemann-Stieltjes integral with $\vartheta(x)$ as the integrator, and integrate by parts:

$$\pi(x) = \int_2^x \frac{d\vartheta(t)}{\log t} = \frac{\vartheta(x)}{\log x} - 1 + \int_2^x \frac{\vartheta(t)}{(\log t)^2}\frac{dt}{t}$$

Thus

$$\left|\pi(x) - \frac{\vartheta(x)}{\log x}\right| \leq 1 + \int_2^x \frac{\vartheta(t)}{(\log t)^2}\frac{dt}{t} \leq 1 + (4\log 2)\int_2^x \frac{dt}{(\log t)^2}\,, \tag{12}$$

where, on the right-hand side, the last inequality is due to Lemma 3.3. We've already seen in (6) that the last integral on the right is $\sim x/(\log x)^2$. From this and (12) we obtain:

$$\left|\pi(x) - \frac{\vartheta(x)}{\log x}\right| = O\left(\frac{x}{(\log x)^2}\right).$$

which, upon multiplying through by $\log x/x$ yields

Note that along with Lemma 3.3, this estimate implies the following weak version of one-half of the PNT:

$$\limsup_{x\to\infty} \pi(x)\frac{\log x}{x} \leq 4\log 2.$$

$$\left|\pi(x)\frac{\log x}{x} - \frac{\vartheta(x)}{x}\right| = O\left(\frac{1}{\log x}\right).$$

In particular:

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} \to 1 \iff \lim_{x\to\infty} \frac{\vartheta(x)}{x} \to 1\,,$$

as promised. □

Metatheorem 3.4. *If the improper integral*

$$\int_1^\infty \left(\frac{\vartheta(t)}{t} - 1\right)\frac{dt}{t} \tag{13}$$

exists, then $\vartheta(x) \sim x$ (which will prove the Prime Number Theorem).

Proof. In the contrapositive spirit, suppose $\vartheta(x)/x$ does not converge to 1 as $x \to \infty$. We wish to show that the improper integral (13) does not exist. There are two possibilities:

(a) There exists $\lambda > 1$ and a sequence $x_n \nearrow \infty$ such that $\vartheta(x_n) \geq \lambda x_n$ for each n, or

(b) There exists $0 < \lambda < 1$ and a sequence $x_n \nearrow \infty$ such that $\vartheta(x_n) \leq \lambda x_n$ for each n.

Suppose we're in situation (a). Then for each n:

$$\int_{x_n}^{\lambda x_n} \left(\frac{\vartheta(t)}{t} - 1\right)\frac{dt}{t} \geq \int_{x_n}^{\lambda x_n} \left(\frac{\vartheta(x_n)}{t} - 1\right)\frac{dt}{t} \geq \int_{x_n}^{\lambda x_n} \left(\frac{\lambda x_n}{t} - 1\right)\frac{dt}{t}$$

Now make the substitution $t = x_n u$ in the last integral to obtain the inequality

$$\int_{x_n}^{\lambda x_n} \left(\frac{\vartheta(t)}{t} - 1\right)\frac{dt}{t} \geq \int_{u=1}^{\lambda} \left(\frac{\lambda}{u} - 1\right)\frac{du}{u}\,,$$

whose right-hand side is positive and independent of n.

Conclusion: For case (a) above, the integral (13) cannot exist.

A similar argument (exercise) works in situation (b). □

4 *The Riemann zeta-function*

So far, purely "real-valued" arguments have shown us that the PNT will follow from the existence of a certain improper integral. The rest of this note serves to illustrate an observation often attributed to Jacques Hadamard:

> ". . . the shortest and best way between two truths of the real domain often passes through the imaginary one."

Relevant to us is the insight of German mathematician Bernhard Riemann who, in the mid 19-th century, realized that the secret to revealing deep truths about the distribution of prime numbers lay in

Georg Friedrich Bernhardt Riemann

holomorphically extending Euler's zeta-function from the real to the complex domain.

In this section we'll show how $\zeta(s)$, defined originally just for the real half-line $(1,\infty)$, extends to a function $\zeta(s)$ that's holomorphic in the *complex* half-plane" $\{\operatorname{Re} s > 1\}$. The key to our proof of the PNT will hinge on making a further holomorphic extension to the punctured right half-plane $\{\operatorname{Re} s > 0\}\backslash\{1\}$, and showing that the zero-free region of this extended zeta-function includes the closed half=plane $\{\operatorname{Re} s \geq 1\}$.

THE RIEMANN ZETA-FUNCTION is defined (initially) on the half-plane $\{\operatorname{Re} s > 1\}$ by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\operatorname{Re} s > 1) \tag{14}$$

Proposition 4.1. *The series on the right-hand side of* (14) *converges absolutely on* $\{\operatorname{Re} s > 1\}$*, and uniformly on each compact subset. Consequently, the function* ζ *so defined is holomorphic in* $\{\operatorname{Re} s > 1\}$*.*

Proof. Writing $s = \sigma + i\tau$ with $\sigma, \tau \in \mathbb{R}$, we know for each $n \in \mathbb{N}$ that $|n^{-s}| = n^{-\sigma}$, so the series in (14) converges absolutely iff $\sigma > 1$. For each $\varepsilon > 0$ and $s \in H_{1+\varepsilon}$, we have $|n^{-s}| < n^{-(1+\varepsilon)}$, so by the Weierstrass M-test, the series in question converges uniformly on the half-plane $H_{1+\varepsilon}$, and in particular, uniformly on each compact subset of H_1. □

THE EULER PRODUCT FORMULA (7) extends to the half-plane $\{\operatorname{Re} s > 1\}$ by essentially the same proof as worked for the real case. The "sieve" argument that produced the formula

$$\prod_{k=1}^{N}\left(1 - \frac{1}{p_k^s}\right)\zeta(s) = 1 + \sum_{p_1,\dots,p_N \nmid n} \frac{1}{n^s} \tag{9}$$

works verbatim for $\operatorname{Re} s > 1$, so continuing with the notation $\Pi_N(s)$ for the product on the left-hand side of (9), and $R_N(s)$ for the sum on the right, we see that

$$|\Pi_N(s)\zeta(s) - 1| \leq |R_N(s)| \leq \sum_{n=p_{N+1}}^{\infty} \frac{1}{|n^s|} = \sum_{n=p_{N+1}}^{\infty} \frac{1}{n^{\operatorname{Re} s}}$$

Since $\operatorname{Re} s > 1$, the series on the right $\to 0$ as $N \to \infty$, hence $\Pi_N(s)\zeta(s)$ converges to 1 as $N \to \infty$. In summary:

Theorem 4.2. *For* $\operatorname{Re} s > 1$*:*

(a) The infinite product $\prod_p \left(1 - \frac{1}{p^s}\right)$ *converges (uniformly on compact sets, in fact) to* $1/\zeta(s)$*, thus proving Euler's formula, and furthermore showing:*

(b) This product is holomorphic, and never takes the value zero there.

From part (b) above:

Corollary 4.3. *The Riemann zeta-function has no zeroes in the half-plane* $\{\mathrm{Re}\, s > 1\}$.

Although the series (14) defining $\zeta(s)$ does not converge in any half-plane $\{\mathrm{Re}\, s > 1 - \varepsilon\}$, a little exercise involving the Riemann-Stieltjes integral shows provides a crucial holomorphic extension. More precisely:

For example, it clearly doesn't converge for s real and < 1. With a little more work, one can show it converges for no complex s with $\mathrm{Re}\, s < 1$ (exercise).

Theorem 4.4 (First extension theorem). $\zeta(s)$ *extends to a function holomorphic on* $\{\mathrm{Re}\, s > 0\}$, *except for a simple pole at* $s = 1$ *where it has residue 1.*

Proof. Recall the notations $[x]$ and $\{x\}$ for the integer and fractional parts of the real number x. Since the integer part function is monotone increasing, it can serve as a Riemann-Stieltjes integrator representing sums. Since it's integrating a continuous function, integration-by-parts works in the usual way[3]. In particular, for $\mathrm{Re}\, s > 1$:

[3] See, e.g., Apostol [2], Chapter 7, esp. Theorem 7.6, page 144].

$$\begin{aligned}
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} &= \int_{1/2}^{\infty} x^{-s} d[x] \\
&= \underbrace{\Big[[x]\, x^{-s}\Big]_{1/2}^{\infty}}_{=0} + s \int_{1/2}^{\infty} [x]\, x^{-s-1}\, dx \\
&= s \int_0^{\infty} (x - \{x\}) x^{-(s+1)}\, dx \\
&= \underbrace{s \int_0^{\infty} x^{-s}\, dx}_{=\frac{s}{s-1} = \frac{1}{s-1} + 1} + s \int_0^{\infty} \{x\} x^{-(s+1)}\, dx
\end{aligned}$$

Since $[x] = 0$ for $0 \le x < 1$, the lower limit in the integral can be any number in the interval $[0, 1)$ that suits us. Here we start with lower limit $1/2$, rather than 0, in order to avoid the appearance of $0/0$ in the "boundary term" produced by integration-by-parts.

Thus,

$$\zeta(s) = \frac{1}{s-1} + H(s) \qquad (\mathrm{Re}\, s > 1), \tag{15}$$

where

$$H(s) := 1 + s \int_0^{\infty} \{x\} x^{-(s+1)}\, dx\,, \tag{16}$$

a function that's holomorphic in the half-plane $\{\mathrm{Re}\, s > 0\}$.[4] It follows that the right-hand side of (15) furnishes a holomorphic extension of $\zeta(s)$ to a function—which we still denote by $\zeta(s)$—to the punctured half-plane $\{\mathrm{Re}\, s > 0\} \backslash \{1\}$, and having a pole at $s = 1$ of order 1, with residue = 1. □

[4] The point here is that the integrand is holomorphic, and its absolute value integrable on $[0, \infty)$ whenever $\mathrm{Re}\, s > 0$. That the integral is holomorphic for $\{s > 0\}$ can be verified, e.g., by using Morera's theorem.

5 *Toward a Proof of the PNT*

We'll continue to use the notation $\zeta(s)$, and the terminology "Riemann zeta-function", but now it will denote the meromorphic continuation given by Theorem 4.4 of the original zeta-function (14) to the "right half-plane" $\{\mathrm{Re}\, s > 0\}$.

We noted early on that—thanks to Euler's Product Formula—the original zeta function never takes on the value zero in its half-plane of definition $\{\operatorname{Re} s > 1\}$. Crucial to our proof of the PNT is the following (seemingly minor) extension of this zero-free region.

Theorem 5.1. $\operatorname{Re} s = 1 \implies \zeta(s) \neq 0$.

Proof. Thanks to Euler's Product Formula (7), we have for $s = \sigma + i\tau$ with $\sigma > 1$:

$$\begin{aligned}\log|\zeta(s)| &= \sum_p \log|1 - p^{-s}|^{-1} \\ &= \sum_p \operatorname{Re}\, \log(1 - p^{-s})^{-1} \\ &= \sum_p \operatorname{Re}\, \sum_{n=1}^{\infty} \frac{p^{-ns}}{n}\end{aligned}$$

so

$$\log|(\zeta(s)| = \sum_p \sum_{n=1}^{\infty} \frac{\cos(n\tau \log p)}{np^{n\sigma}}. \tag{17}$$

It follows that upon letting $\vartheta = \tau \log p$, we have ("out of a hat")

$$\begin{aligned}&\log|\zeta(\sigma))^3\zeta(\sigma + i\tau)^4\zeta(\sigma + 2i\tau)| \\ &\quad = \sum_p \sum_{n=1}^{\infty} \frac{3 + 4\cos(n\vartheta) + \cos(2n\vartheta)}{np^{n\sigma}} \\ &\quad = \sum_p \sum_{n=1}^{\infty} \frac{2 + 4\cos(n\vartheta) + 2\cos^2(n\vartheta)}{np^{n\sigma}} \\ &\quad = \sum_p \sum_{n=1}^{\infty} \frac{2(1 + \cos(n\vartheta)^2}{np^{n\sigma}}\end{aligned}$$

Since each of the final summands is positive, so is

$$\log|\zeta(\sigma))^3\zeta(\sigma + i\tau)^4\zeta(\sigma + 2i\tau)|$$

which, upon exponention, yields:

$$|\zeta(\sigma)^3\zeta(\sigma + i\tau)^4\zeta(\sigma + 2i\tau)| \geq 1. \tag{18}$$

Note that we have, at this point, another proof that the Riemann zeta-function has no zeros in its original domain of definition H_1.

whenever $s = \sigma + i\tau$ with $\sigma > 1$, and—by continuity—even for $\sigma = 1$.

Now suppose, for the sake of contradiction, that $\zeta(1 + i\tau_0) = 0$ for some $\tau_0 \in \mathbb{R}$. Then $\tau_0 \neq 0$ since (our analytically extended) zeta-function has a pole at $s = 1$. Now consider the limit as $\sigma \to 1+$ of

$$f(\sigma) := \lim_{\sigma \to 1+} \zeta(\sigma)^3\zeta(\sigma + i\tau_0)^4\zeta(\sigma + 2i\tau_0).$$

We know from Proposition 4.4 that $\zeta(s)^3$ has a pole of order 3 at $s = 1$, and from hypothesis we know that $\zeta(s)^4$ has a zero of order at least 4 at $s = 1 + i\tau_0$. Thus the product of the first two terms on the right-hand side of the above definition of f has limit 0 as $\sigma \to 1+$.

Since ζ extends holomorphically to $H_0\backslash\{1\}$, it doesn't have a pole at $1+2i\tau_0$, hence $\lim_{\sigma\to 1-} f(\sigma)=0$, contradicting our earlier observation (18) that $|f(\sigma)|\ge 1$ for $\sigma>1$.

This completes our proof that the (extended) zeta-function never assumes the value zero on the line $\operatorname{Re} s=1$. □

We're now in a position to see how to prove the Prime Number Theorem in the form "$\vartheta(x)\sim x$" (see page 6). The proof will depend on:

- Euler's formula (2.1) for $\zeta(s)$;
- Metatheorem 3.4, showing that "$\theta(x) \sim x$" follows from the existence of a certain improper integral;
- Theorem 5.1, providing the new information that $\zeta(s)$ has no zero on the line $\operatorname{Re} s=1$, and finally;
- A "Tauberian Theorem" to be proved for Laplace transforms.

Recall that, thanks to to Euler's formula

$$\zeta(s)=\prod_p (1-p^{-s})^{-1} \qquad (Res>1),$$

we know that $\zeta(s)$ is holomorphic and non-zero on $H_1:=\{\operatorname{Re} s>1\}$. Thus $\log(\zeta(s)$ is also holomorphic on H_1, and its derivative there is

$$\frac{\zeta'(s)}{\zeta(s)}=\sum_p \frac{d}{ds}(1-p^{-s})^{-1}=-\sum_p \frac{p^{-s}\log p}{1-p^{-s}}=-\sum_p \frac{\log p}{p^s-1} \tag{19}$$

We memorialize this calculation in

Definition 5.2.

$$Z(s):=-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s-1} \qquad (\operatorname{Re} s>1).$$

Since $\zeta(s)\neq 0$ at any point of H_1, we know that $Z(s)$ is holomorphic there. Since we've been able to extend $\zeta(s)$ meromorphically to $\{\operatorname{Re} s>0\}$, the same is true for $Z(s)$, with each pole or zero of $\zeta(s)$ providing a simple pole for $Z(s)$. In particular,

> *$Z(s)$ is meromorphic $\{\operatorname{Re} s>0\}$ and holomorphic in $\{\operatorname{Re} s\ge 1\}$, except for a pole of order 1 at $s=1$, for which the corresponding residue is 1.*

Equation (19) suggests a connection between $Z(s)$ and the Chebyshev function $\vartheta(x)=\sum_{p\le x}\log p$. A preliminary calculation makes this a bit more explicit.

$$Z(s) = \underbrace{\sum_p \frac{\log p}{p^s}}_{:=\Phi(s)} + \underbrace{\sum_p \frac{\log p}{p^s(p^s-1)}}_{:=H(s)} \qquad (\operatorname{Re} s>1) \tag{20}$$

Here $H(s)$ extends holomorphically to $H_{1/2} = \{\mathrm{Re}\, s > 1/2\}$, so at least in that half-plane the function $\Phi(s)$ is meromorphic, with the same poles as $Z(s)$; in particular:

> *$\Phi(s)$ is meromorphic in $\{\mathrm{Re}\, s > 1/2\}$ and holomorphic there, except for a pole of order 1 at $s = 1$, for which the corresponding residue is 1.*

Now $\vartheta(x)$ increases monotonically on $[1, \infty)$, so it can serve as a Riemann-Stieltjes integrator, thus allowing the definition of $\Phi(s)$ to be recast as a (Riemann-Stieltjes) integral:

$$\Phi(s) = \int_1^\infty x^{-s}\, d\vartheta(x) \qquad (\mathrm{Re}\, s > 1)$$

Integration-by-parts works as well for Riemann-Stieltjes integrals as for ordinary ones, yielding in this case

$$\Phi(s) = \left[\frac{\vartheta(x)}{x^s}\right]_{x=1}^\infty + s\int_1^\infty x^{-(s+1)}\vartheta(x)\, dx \qquad (\mathrm{Re}\, s > 1).$$

The "boundary terms" vanish since by definition $\vartheta(1) = 0$ and, by Theorem 3.3, $\vartheta(x) = O(x)$ as $x \to \infty$. Thus we have $\Phi(s)$ expressed as a "Mellin Transform" of ϑ:

(21) $$\Phi(s) = s\int_1^\infty x^{-(s+1)}\vartheta(x)\, dx \qquad (\mathrm{Re}\, s > 1)$$

Now recall:

(a) From Metatheorem 3.2 that we can prove the Prime Number Theorem by showing that $\vartheta(x) \sim x$ as $x \to \infty$, and that

(b) by Metatheorem 3.4, to show this it suffices to prove the existence of the improper integral

(22) $$\int_1^\infty \left(\frac{\vartheta(x)}{x} - 1\right)\frac{dx}{x} := \lim_{A\to\infty}\int_1^A \left(\frac{\vartheta(x)}{x} - 1\right)\frac{dx}{x}.$$

Toward this goal, let's express the right-hand side of (21) in terms of $\dfrac{\vartheta(x)}{x} - 1$:

Note that $\frac{\vartheta(x)}{x} - 1$ is the relative error committed in the approximation of $\vartheta(x)$ by x.

$$\begin{aligned}\Phi(s) &= s\int_1^\infty (\vartheta(x) - x)x^{-(s+1)}\, dx + s\int_1^\infty x^{-s}\, dx \\ &= s\int_1^\infty \left(\frac{\vartheta(x)}{x} - 1\right)x^{-s}\, dx + \frac{s}{s-1}\end{aligned}$$

Thus our representation (20) of $Z(s) = -\zeta'(s)/\zeta(s)$ can be rewritten for $\mathrm{Re}\, s > 1$:

(23) $$Z(s) = \frac{1}{s-1} + 1 + s\int_1^\infty \left(\frac{\vartheta(x)}{x} - 1\right)x^{-s}\, dx + H(s).$$

where the function H is holomorphic in $\{\mathrm{Re}\, s > 1/2\}$. Moreover recall that:

(a) We've extended the zeta-function holomorphically to the "punctured" open right half-plane $\{\mathrm{Re}\, s > 0\}\backslash\{0\}$, noting that this extended function has a pole at $s = 1$ for which the residue is 1, and that it never vanishes on the rest of the vertical line $\{\mathrm{Re}\, s = 1\}$

(b) Therefore $Z = -\zeta'/\zeta$ is holomorphic in the open half-plane $\{\mathrm{Re}\, s > 0\}$, except for any poles or zeros that ζ may have there, and that, none of these lie on the "punctured line" $\{\mathrm{Re}\, s = 1\}\backslash\{1\}$.

(c) Each pole (resp. zero) of ζ bestows upon Z a simple pole with residue equal to the order of the pole (resp. minus the order of the zero).

(d) We've shown in our proof of Euler's formula (7) that $\zeta(s) \neq 0$ whenever $\mathrm{Re}\, s$ is > 1, and in Theorem 5.1 have extended this result to $\mathrm{Re}\, s = 1$. Consequently,

Z is holomorphic in (an open set containing) the punctured closed half-plane $\{\mathrm{Re}\, s \geq 1\}\backslash\{1\}$. Moreover Z has a simple pole at $s = 1$, with residue equal to 1.

Because of item (d) in the above list, we have:

Theorem 5.3. *The integral*

$$F(s) := \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) x^{-s}\, dx$$

extends holomorphically from $\{\mathrm{Re}\, s > 1\}$ to an open set containing the closed half-plane $\{\mathrm{Re}\, s \geq 1\}$.

Proof. We see this from rewriting (23), for $\mathrm{Re}\, s > 1$, as

$$F(s) = \frac{1}{s}\left(Z(s) - \frac{1}{s-1} - 1 - H(s) \right).$$

Since Z is holomorphic in an open set containing $\{\mathrm{Re}\, s \geq 1\}$ except for a simple pole with residue 1 at $s = 1$, and since H is holomorphic for $\mathrm{Re}\, s > 1/2$, the right-hand side of this equation is holomorphic in an open set containing $\{\mathrm{Re}\, s \geq 1\}$, and so furnishes the desired extension of F. □

In particular, upon restricting s to real values, we obtain:

Corollary 5.4.

$$\lim_{s \to 1+} \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) x^{-s}\, dx$$

exists (and $= F(1)$).

OUR GOAL of proving that the improper integral

$$\int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) \frac{dx}{x} \tag{13}$$

exists can be stated like this:

For $1 \le x < \infty$ and $A > 1$, let $\chi_A(x) = 0$ if $x > A$, and $= 0$ otherwise. To say "the improper integral (13) exists" means that the (finite) limit:

$$\lim_{A\to\infty} \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1\right) \chi_A(x) \frac{dx}{x}$$

exists.

I.e., let χ_A be the characteristic function of the interval $[1, A]$.

By contrast, Corollary 5.4 asserts that a *different* kind of improper integral really *does* exist; this one defined by the limiting process

$$\lim_{s\to 1+} \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1\right) x^{1-s} \frac{dx}{x}$$

where the convergence factor $\chi_A(x)$ used to define the original improper integral (13) is replaced by a new one: x^{1-s}.

So to prove the Prime Number Theorem we need to show that the existence of this new type of improper integral implies the existence of the usual one. This will occupy our efforts for the next two sections.

6 *Newman's Tauberian Theorem*

At this point it's convenient to make the change-of variable $x = e^t$ in the integrals we're studying. Thus, upon writing

$$g(t) := \frac{\vartheta(e^t)}{e^t} - 1\,,$$

the original improper integral, whose existence we wish to establish, becomes

$$\int_0^\infty g(t)\,dt := \lim_{A\to\infty} \int_0^A g(t)\,dt\,,$$

while the modified improper integral, whose existence we *have* established, is

$$\lim_{s\to 1-} \int_0^\infty g(t) e^{-(s-1)t}\,dt\,.$$

where this last integral is the Laplace transform $\mathscr{L}\{g\}(s-1)$.

The *Laplace transform* of a function $f : [0, \infty) \to \mathbb{C}$ that is locally integrable and (say) bounded is defined for $\operatorname{Re} s > 0$ by

$$\mathscr{L}\{f\}(s) = \int_0^\infty f(t) e^{-st}\,dt$$

In summary: Thanks to Theorem 5.3 we know that $\mathscr{L}\{g\}(s-1)$ is holomorphic in a neighborhood of $\{\operatorname{Re} s \ge 1\}$, hence

$\mathscr{L}\{g\}(s)$ is holomorphic in a neighborhood of $\{\operatorname{Re} s \ge 0\}$.

Thus $\lim_{s\to 0+} \mathscr{L}\{g\}(s)$ exists (and $= \mathscr{L}\{g\}(0)$). Our task now is to show that the improper integral $\int_0^\infty g(t)\,dt$ exists in the usual sense—and equals $\mathscr{L}\{g\}(0)$. This (and with it the Prime Number Theorem) will follow immediately from:

Theorem 6.1 (Newman's Tauberian Theorem)**.** *Suppose $f : [0, \infty) \to \mathbb{C}$ is bounded and locally integrable. If its Laplace transform*

$$\mathscr{L}\{f\}(s) := \int_0^\infty f(t) e^{-st}\,dt \qquad (\operatorname{Re} s > 0)$$

extends holomorphically to a neighborhood of the closed half-plane $\{\mathrm{Re}\, s \geq 0\}$*, then the improper integral* $\int_0^\infty f(t)\,dt$ *exists (and equals* $\mathscr{L}\{f\}(0)$*).*

The proof of Newman's theorem, while not trivial, depends only on the Cauchy Integral Theorem of elementary complex analysis. It will occupy Section 7 below. However before presenting this argument, let's take some time to understand the meaning of "tauberian".

A Tauberian Interlude A *tauberian theorem* is, roughly speaking, one that shows how "convergence in some weak sense" plus some extra condition implies "convergence in the ordinary sense." The terminology comes from:

Theorem 6.2 (Tauber's Theorem, 1897)**.** *Suppose the power series* $\sum_{n=0}^\infty a_n z^n$ *has radius of convergence* 1*, and that* $lim_{r\to 1-} \sum_{n=0}^\infty a_n r^n = A$*. If, in addition,* $\lim_{n\to\infty} na_n = 0$*, then* $\sum_{n=0}^\infty a_n = A$*.*

Here the convergence of $\sum_{n=0}^\infty a_n$ is helped by the insertion of "convergence factors" r^n. Tauber's Theorem then asserts that the "Tauberian condition" $a_n = o(1/n)$ renders these helper-factors unnecessary. If the series-coefficients a_n are all ≥ 0, the tauberian condition can be weakened to just: $a_n \to 0$. In 1910 Littlewood the tauberian condition of Tauber's theorem from $|a_n| = o(n)$ to $a_n = O(n)$.

An earlier result of Abel goes in the other direction:

Theorem 6.3 (Abel's Theorem)**.** $\displaystyle\sum_{n=0}^\infty a_n = A \implies \lim_{r\to 1-} \sum_{n=0}^\infty a_n r^n = A.$

The proof is a fairly simple argument based on summation-by-parts.

We call a sequence (a_n) *Abel summable (to* $A \in \mathbb{C}$ if $\lim_{r\to 1-} a_n r^n = A$. Abel's theorem asserts that any sequence that's summable in the ordinary sense is also Abel summable. However, the Abel method creates new summable sequences. For example, if $a_n = (-1)^n$ then the series $\sum_{n=0}^\infty a_n$ diverges in the usual sense, but, thanks to the Geometric Series Theorem, Abel-sums to 1/2.

Results of this sort—asserting that ordinary convergence survives the insertion of "helpers"—are called "abelian theorems". Here, for example, is an abelian companion to Newman's Tauberian Theorem.

Theorem 6.4 (An abelian theorem for the Laplace transform)**.**

$$\lim_{A\to\infty} \int_0^A f(t)\,dt = A \implies \lim_{s\to 0+} \int_0^\infty f(t)e^{-st}\,dt = A.$$

Proof. We subject the integral of interest to an integration by parts. Let $F(t) = \int -0^t f(x)\,dx$, so

$$\begin{aligned}
\int_0^\infty f(t)e^{-st}\,dt &= \int_0^\infty e^{-st}\,dF(t) \\
&= \underbrace{\Big[e^{-st}F(t)\Big]_{t=0}^\infty}_{=0} + s\int_0^\infty F(t)e^{-st}\,dt \\
&= \int_{x=0}^\infty F(x/s)e^{-x}\,dx
\end{aligned}$$

Substitute $x = st$ in the last integral.

Now $F(x/s) \to A$ boundedly as $s \to 0+$, so by the Lebesgue Dominated Convergence Theorem:

$$\lim_{s\to 0+} \int_0^\infty F(x/s)e^{-x}\,dx = A\int_0^\infty e^{-x}\,dx = A. \qquad \square$$

We close this little interlude with an example that shows how Newman's theorem can fail if the holomorphic-extendability condition is weakened.

Example 6.5. For n a non-negative integer, Let χ_n be the characteristic function of the interval $[n, n+1)$, and set $f = \sum_{n=0}^\infty (-1)^n \chi_n$, so $f(t) = (-1)^n$ for $t \in I_n$. Then for $\operatorname{Re} s > 0$:

$$\begin{aligned}\mathscr{L}\{f\}(s) &= \int_0^\infty f(t)e^{-st}\,dt = \sum_{n=0}^\infty (-1)^n \int_n^{n+1} e^{-st}\,dt \\ &= \left(\sum_{n=0}^\infty (-1)^n e^{-ns}\right)\frac{1-e^{-s}}{s}.\end{aligned}$$

Thus, thanks to the geometric series theorem,

$$\mathscr{L}\{f\}(s) = \frac{1}{1+e^{-s}}\frac{1-e^{-s}}{s} \qquad (\operatorname{Re} s > 0). \tag{24}$$

Consequently $\mathscr{L}\{f\}(s) \to 1/2$ as $s \to 0+$, so—while $\int_0^\infty f(s)\,ds$ does not exist in the usual sense, it *does* exist in the sense of the Newman-theorem hypotheses (i.e., when the "helper" e^{-st} multiplies f).

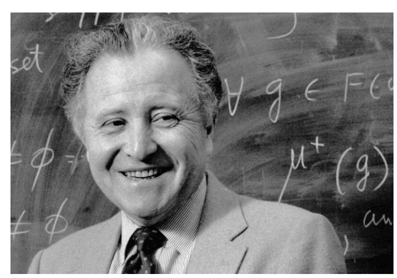
The problem is, of course, that $\mathscr{L}\{f\}$ does not extend holomorphically across the imaginary axis. To see this, let's rewrite (24) as

$$\mathscr{L}\{f\}(s) = \frac{H(s)}{1+e^{-s}} \qquad (\operatorname{Re} s > 0),$$

where $H(s) = s^{-1}(1-e^{-s})$ is an entire function[5] that takes the value zero only at the even integer multiples of πi on the imaginary axis. Thus $\mathscr{L}\{f\}$ has a pole on the imaginary axis at each *odd* integer multiple of πi on the imaginary axis, and so is not holomorphically extendable across that whole axis (although, note that it *does* extend across many open subintervals, e.g., $(-\pi i, \pi i)$).

[5] *Proof.* Expand it in a power series.

7 *Proof of Newman's Tauberian Theorem*



D. J. Newman, 1930-2007

In this section we prove Newman's Tauberian Theorem, and thus finish the proof of the Prime Number Theorem.

Recall that we are given a function $f\colon [0,\infty) \to \mathbb{C}$ that is both bounded and locally integrable on $[0,\infty)$, so that we may form its *Laplace transform*

$$\mathscr{L}\{f\}(z) := \int_0^\infty f(t)e^{-zt}\,dt \qquad (\operatorname{Re} z > 0)\,.$$

In this section we switch from using the notation $s = \sigma + i\tau$ for a generic complex number to the more usual (for complex analysts) $z = x + iy$.

Thus $\mathscr{L}\{f\}$ is holomorphic on the right half-plane $H_0 = \{\operatorname{Re} z > 0\}$. Newman's Theorem asserts that:

Under the additional assumption that $\mathscr{L}\{f\}$ is holomorphic on the closed *half-plane* $\{\operatorname{Re}\{z\} \geq 0\}$, *the improper integral*

$$\int_0^\infty f(t)\,dt := \lim_{T\to\infty} \int_0^T f(t)\,dt$$

exists (and equals $\mathscr{L}\{f\}(0)$).

To say a function is holomorphic on a *closed* set F means that it's holomorphic on an open set that contains F.

Proof. For $\operatorname{Re} z > 0$, set

$$g(z) = \mathscr{L}\{f\}(z) = \int_0^\infty f(t)e^{-zt}\,dt$$

so g is holomorphic in an open set V containing the closed right half-plane, and for $T > 0$ set

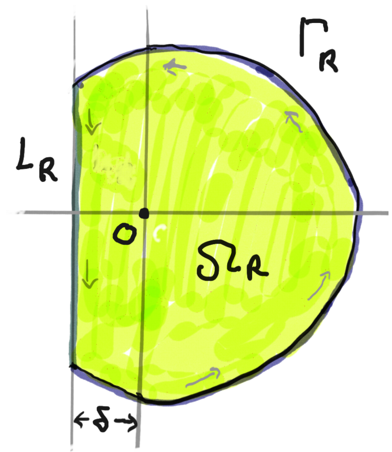
$$g_T(z) := \mathscr{L}\{f\chi_{[0,T]}\}(z) = \int_0^T f(t)e^{-zt}\,dt\,.$$

so g_T is an entire function (holomorphic on $\mathbb{C}$).

To Show: $\lim_{T\to 0+} g_T(0) = g(0)$:

(a) *First observation.* For each $R > 0$ there exists $\delta = \delta_R > 0$ such that the closed set $\Omega_R := \{|z| < R\} \cap \{\operatorname{Re} z > -\delta\}$ lies in V.

Thus C_R, the positively oriented boundary of Ω_R, is the union of is the circular segment $\Gamma_R := \{|z| = R\} \cap \{\operatorname{Re} z \geq \delta\}$, and a segment L_R, of the line $\{\operatorname{Re} z = \delta\}$.Thus C_R is a simple, closed, piecewise differentiable curve in V that contains the origin, so by the Cauchy integral theorem:



$C_R = \partial\Omega = \Gamma_R \cup L_R$

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_{C_R} [g(z) - g_T(z)]\frac{dz}{z}\,.$$

While true, this representation of $g(0) - g_T(0)$ doesn't quite do the trick. What's needed is:

(b) *Newman's modification.* For $R, T > 0$ and $z \in \mathbb{C}$ set

$$W_{T,R}(z) := e^{zT}\left(1 + \frac{z^2}{R^2}\right).$$

Thus $W_{T,R}$ is an entire function with $W_{T,R}(0) = 1$, so

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_{C_R} [g(z) - g_T(z)]W_{T,R}(z)\frac{dz}{z}\,. \tag{25}$$

We estimate part of this the integral over the circular segment Γ_R. For $z = x + iy \in \Gamma_R$ we have

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}\right| \leq \|f\|_\infty \int_0^T |e^{-zt}|\,dt$$
$$= \|f\|_\infty \int_0^T e^{-xt}\,dt = \|f\|_\infty \frac{e^{-Tx}}{x}\,.$$

Now for any complex number z with $|z| = R$ we have

$$\left|1 + \frac{z^2}{R^2}\right| = \frac{2\operatorname{Re} z}{R}. \tag{26}$$

In view of (26) and the estimate above it, we see that for $z = x + iy \in \Gamma_R$ the integrand on the right-hand side of (25) is, in absolute value:

$$|g(z) - g_T(z)|\frac{|W_{T,R}(z)|}{|z|} \leq \|f\|_\infty \frac{e^{-Tx}}{x} \cdot e^{Tx} \cdot \frac{2x}{R} \cdot \frac{1}{R} = \frac{2\|f\|_\infty}{R^2},$$

Here x is sometimes negative!

from which we conclude (taking into account that the length of C_R^+ is $< 2\pi R$):

$$\left|\int_{\Gamma_R} [g(z) - g_T(z)]\, W_{T,R}(z)\frac{dz}{z}\right| \leq \frac{4\pi\|f\|_\infty}{R}. \tag{27}$$

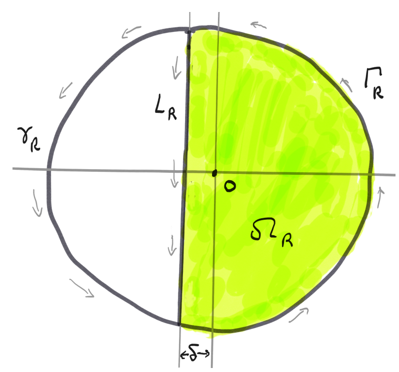
Observe that this estimate is independent of T. (!!)

(c) *The integral over L_R*. We have L_R parameterized by $z = -\delta + iy$ for $|y| < \text{length}(L_R)/2 < R$. Thus we can estimate the integral involving g over L_R as follows.

Important to note that we don't have an explicit formula for g on L_R; we just know that it's the analytic continuation of g from $\{\operatorname{Re} s > 0\}$ to some open set containing $\{\operatorname{Re} s \geq 0\}$.

$$\left|\int_{L_R} g(z)W_{T,R}(z)\frac{dz}{z}\right| \leq e^{-\delta T} \underbrace{\int_{L_R} |g(z)| \left|1 + \frac{z^2}{R^2}\right| \frac{1}{|z|}\, |dz|}_{:=M(R),\ \text{independent of } T}$$
$$= M(R)e^{-\delta T}$$

(d) *The integral involving g_T*. The integrand here is an entire function of z, so by the Cauchy integral theorem we can replace the line segment L_R by the circular arc $\gamma_R = \{|z| = R\} \cap \{\operatorname{Re} z \leq -\delta\}$, after which the estimate proceeds as in part (b). For $z = x + iy \in \gamma_R$



$\Gamma_R \cup \gamma_R = \{|z| = R\}$

$$|g_T(z)| \leq \int_0^T |f(t)||e^{-zt}|\, dt \leq \|f\|_\infty \int_0^T e^{-xt}\, dt = \|f\|_\infty \frac{e^{-xT}}{|x|}.$$

Thus, as in part (b):

$$\left|\int_{\gamma_R} g_T(z)W_{T,R}(z)\frac{dz}{z}\right| \leq \|f\|_\infty \cdot \frac{e^{-xT}}{x} \cdot e^{xT} \cdot \frac{2x}{R^2} \cdot \pi R$$
$$\leq \frac{2\pi\|f\|_\infty}{R}.$$

Note that here, x is *always* negative!

Putting it all together: for each $R, T > 0$:

$$|g(0) - g_T(0)| \leq \frac{2\|f\|_\infty}{R} + \frac{M(R)}{2\pi}e^{-\delta T} + \frac{\|f\|_\infty}{R}.$$

Don't forget that we have to divide all the integrals by 2π in using them to calculate $g(0) - g_T(0)$.

Let $\varepsilon > 0$ be given. Choose R to make $\frac{3\pi\|f\|_\infty}{R} < \varepsilon/2$, and having done this, choose T_0 to make $\frac{M(R)}{2\pi}e^{-\delta T_0} < \varepsilon/2$. Then for all $T \geq T_0$ we have

$$|g(0) - g_T(0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\lim_{T\to\infty} g_T(0) = g(0)$, as desired. □

8 Epilogue

Our proof of the PNT hinged on showing that the Riemann zeta-function doesn't vanish on the line $\{\operatorname{Re} s = 1\}$. It turns out that this non-vanishing-ness is yet another property *equivalent* to the PNT.

Metatheorem 8.1. *PNT* $\implies \zeta(s) \neq 0$ *whenever* $\operatorname{Re} s = 1$.

Proof. We're assuming the PNT, but not the never-zero-ness of ζ on $\{\operatorname{Re} s = 1\}$. Suppose, for the sake of contradiction, that $\zeta(s_0) = 0$ for some point $s_0 = 1 + i\tau_0$ with τ_0 real and (necessarily) $\neq 0$. Thus Z has a simple pole at s_0 with residue $m > 0$, the order of the ζ's supposed zero at s_0. Thus, $m = \lim_{s \to s_0}(s - s_0)Z(s)$, so by (23) (and the fact that H is holomorphic in $\{\operatorname{Re} s > 1/2\}$, hence bounded in a neighborhood of s_0) we see that

$$m = \lim_{s \to s_0} (s - s_0) \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) x^{-s}\, dx \tag{28}$$

Now we're assuming that the PNT has been proved, i.e., in view of Metatheorem 3.2, that $\vartheta(x) \sim x$. Consequently, given $\varepsilon > 0$ we can choose $X = X(\varepsilon) > 1$ such that

$$\left| \frac{\vartheta(x)}{s} - 1 \right| < \varepsilon \quad \forall\, x > X,$$

hence for $s = \sigma + i\tau$, with $\sigma > 1$: (upon noting that $|x^{-s}| = x^{-\sigma}$):

$$\begin{aligned} \left| \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) x^{-s}\, dx \right| &\leq \int_1^\infty \left| \frac{\vartheta(x)}{s} - 1 \right| x^{-\sigma}\, dx \\ &\leq \underbrace{\int_1^X \left| \frac{\vartheta(x)}{s} - 1 \right| x^{-\sigma}\, dx}_{=:M(\varepsilon)<\infty} + \varepsilon \underbrace{\int_X^\infty x^{-\sigma}\, dx}_{=\frac{1}{\sigma-1}} . \end{aligned}$$

Thus

$$\limsup_{\sigma \to 1+} (\sigma - 1) \left| \int_1^\infty \left(\frac{\vartheta(x)}{x} - 1 \right) x^{-s}\, dx \right| \leq \varepsilon.$$

Since ε can be any positive number, this last line asserts that if s is restricted to the line $\{\operatorname{Im} s = \tau_0\}$, then the limit, m, on the right-hand side of (28) is zero, contradicting the fact that $m > 0$. Thus PNT implies that $\zeta(s) \neq 0$ whenever $\operatorname{Re} s = 1$. □

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